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ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY,

WITH

NUMEROUS EXAMPLES.

BY

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BOSTON, U.S.A.:
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PREFACE.

THE present work on Plane and Spherical Trigonometry is designed as a text-book for High Schools, Academies, and Colleges. It is a brief practical course, intended for those who wish to learn the elements of Trigonometry, which subject it covers as far as to the end of the solution of Spherical triangles. The aim has been to give simple and elementary demonstrations, to make the subject clear and interesting to the learner, and to furnish only so much of Trigonometry as is actually taught in our best Colleges.

The examples are very numerous and for the most part easy, and are carefully selected, so as to illustrate every part of the subject. Those at the ends of the chapters are progressive, and arranged so as to give the student a thorough drill in trigonometric work and in numerical computation. Especial attention has been given to the numerical solutions of triangles, each case being treated in detail.

It is assumed in this work that the student has received from Algebra a knowledge of the nature and properties of logarithms, sufficient for practical application to the solution of triangles by means of logarithmic tables.

My thanks are due to those of my friends who have kindly assisted me in reading the MS. and correcting the proof-sheets.

E. A. B.

RUTGERS COLLEGE,
New Brunswick, N.J., May, 1892.

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ELEMENTS OF TRIGONOMETRY.

PART I.

PLANE TRIGONOMETRY.

CHAPTER I.

MEASUREMENT OF ANGLES.

1. Trigonometry is that branch of mathematics which treats (1) of the solution of plane and spherical triangles, and (2) of the general relations of angles and certain functions of them called the *trigonometric functions.*

Plane Trigonometry comprises the solution of plane triangles and investigations of plane angles and their functions.

Trigonometry was originally the science which treated only of the sides and angles of plane and spherical triangles; but it has been recently extended so as to include the analytic treatment of all theorems involving the consideration of angular magnitudes.

2. The Measure of a Quantity. — All measurements of lines, angles, etc., are made in terms of some fixed standard or *unit,* and *the measure of a quantity* is the number of times the quantity contains the unit.

It is evident that the same quantity will be represented by different numbers when different units are adopted. For example, the distance of a mile will be represented by the number 1 when a mile is the unit of length, by the number 1760 when a yard is the unit of length, by the number 5280 when a foot is the unit of length, and so on. In like manner, the number expressing the magnitude of an angle will depend on the unit of angle.

EXAMPLES.

1. What is the measure of $2\frac{1}{2}$ miles when a yard is the unit?

$$2\tfrac{1}{2} \text{ miles} = \tfrac{5}{2} \times 1760 \text{ yards}$$
$$= 4400 \text{ yards} = 4400 \times 1 \text{ yard.}$$

$\therefore$ the measure is 4400 when a yard is the unit.

2. What is the measure of a mile when a chain of 66 feet is the unit? *Ans.* 80.

3. What is the measure of 2 acres when a square whose side is 22 yards is the unit? *Ans.* 20.

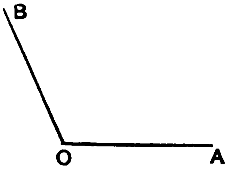
4. The measure of a certain field is 44 and the unit is 1100 square yards; express the area of the field in acres. *Ans.* 10 acres.

5. If 7 inches be taken as the unit of length, by what number will 15 feet 2 inches be represented? *Ans.* 26.

6. If 192 square inches be represented by the number 12, what is the unit of linear measurement? *Ans.* 4 inches.

3. Angles. — *An angle* is the opening between two straight lines drawn from the same point. The point is called the *vertex* of the angle, and the straight lines are called the *sides* of the angle.

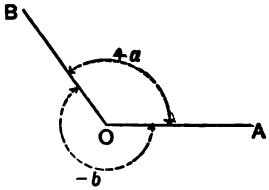
An angle may be generated by revolving a line from coincidence with another line about a fixed point. The initial and final positions of the line are the sides of the angle; the amount of revolution measures the magnitude of the angle; and the angle may be traced out by any number of revolutions of the line.



Thus, to form the angle AOB, OB may be supposed to have revolved from OA to OB; and it is obvious that OB may go on revolving until it comes into the same position OB as many times as we please; the angle AOB, having the same bounding lines OA and OB, may therefore be greater than 2, 4, 8, or any number of right angles.

The line OA from which OB moves is called the *initial line*, and OB in its final position, the *terminal line*. The revolving line OB is called the *generatrix*. The point O is called the *origin*, *vertex*, or *pole*.

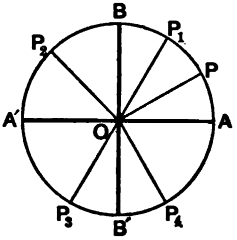
4. Positive and Negative Angles. — We supposed in Art. 3 that OB revolved in the direction opposite to that of the hands of a watch. But angles may, of course, be described by a line revolving in the same direction as the hands of a watch, and it is often necessary to distinguish between the two directions in which angles may be measured from the same fixed line. This is conveniently effected by adopting the convention that angles measured in one direction shall be considered positive, and angles measured in the opposite direction, negative. In all branches of mathematics angles described by the revolution of a straight line in the direction *opposite* to that in which the hands of a watch move are usually considered *positive*, and all angles described by the revolution of a straight line in the *same* direction as the hands of a watch move are considered *negative*.



Thus, the revolving line OB starts from the initial line OA. When it revolves in the direction *contrary* to that of the hands of a watch, and comes into the position OB, it traces out the *positive* angle AOB (marked $+a$); and when it revolves in the *same* direction as the hands of a watch, it traces the *negative* angle AOB (marked $-b$).

The revolving line is always considered positive.

5. The Measure of Angles. — An angle is measured by the arc of a circle whose centre is at the vertex of the angle and whose ends are on the sides of the angle (Geom. Art. 236).



Let the line OP of fixed length generate an angle by revolving in the positive direction round a fixed point O from an initial position OA. Since OP is of constant length, the point P will trace out the circumference ABA′B′ whose centre is O. The two perpendicular diameters AA′ and BB′ of this circle will inclose the four right angles AOB, BOA′, A′OB′, and B′OA.

The circumference is divided at the points A, B, A′, B′ into four *quadrants*, of which

AB is called the *first quadrant.*
BA′ " " " *second quadrant.*
A′B′ " " " *third quadrant.*
B′A " " " *fourth quadrant.*

In the figure, the angle AOP_1, between the initial line OA and the revolving line OP_1, is less than a right angle, and is said to be *an angle in the first quadrant.* AOP_2 is greater than one and less than two right angles, and is said to be *an angle in the second quadrant.* AOP_3 is greater than two and less than three right angles, and is said to be *an angle in the third quadrant.* AOP_4 is greater than three and less than four right angles, and is said to be *an angle in the fourth quadrant.*

When the revolving line returns to the initial position OA, the angle AOA is an angle of four right angles. By supposing OP to continue revolving, the angle described will become greater than an angle of four right angles. Thus, when OP coincides with the lines OB, OA′, OB′, OA, in the second revolution, the angles described, measured from the beginning of the first revolution, are angles of five right angles, six right angles, seven right angles, eight right angles, respectively, and so on. By the continued revolution of OP the angle between the initial line OA and the revolving line OP may become of any magnitude whatever.

In the same way OP may revolve in the *negative* direction about O any number of times, generating a *negative* angle; and this negative angle may obviously have any magnitude whatever.

The angle AOP may be the geometric representative of any of the Trigonometric angles formed by any number of complete revolutions, either in the *positive* direction added to the *positive* angle AOP, or in the *negative* direction added to the *negative* angle AOP. In all cases the angle is said to be in the quadrant indicated by its terminal line.

There are three methods of measuring angles, called respectively the Sexagesimal, the Centesimal, and the Circular methods.

6. The Sexagesimal Method. — This is the method in general use. In this method the right angle is divided into 90 equal parts, each of which is called a *degree.* Each degree is subdivided into 60 equal parts, each of which is called a *minute.* Each minute is subdivided into 60 equal parts, each of which is called a *second.* Then the magnitude of an angle is expressed by the number of degrees, minutes, and seconds which it contains. Degrees, minutes, and seconds are denoted respectively by the symbols $^\circ$, $'$, $''$: thus, to represent 18 degrees, 6 minutes, 34.58 seconds, we write

$$18^\circ\ 6'\ 34''.58.$$

A degree of arc is $\frac{1}{360}$ of the circumference to which the arc belongs. The degree of arc is subdivided in the same manner as the degree of angle.

Then $\quad 1 \text{ circumference} = 360^\circ = 21600' = 1296000''.$
$\qquad\quad 1 \text{ quadrant or right angle} = 90^\circ.$

Instruments used for measuring angles are subdivided accordingly.

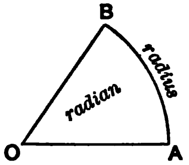
7. The Centesimal or Decimal Method. — In this method the right angle is divided into 100 equal parts, each of which is called a *grade*. Each grade is subdivided into 100 equal parts, each of which is called a *minute*. Each minute is subdivided into 100 equal parts, each of which is called a *second*. The magnitude of an angle is then expressed by the number of grades, minutes, and seconds which it contains. Grades, minutes, and seconds are denoted respectively by the symbols ^{g}, $^{\backprime}$, $^{\backprime\backprime}$: thus, to represent 34 grades, 48 minutes, 86.47 seconds, we write

$$34^{g}\,48^{\backprime}\,86^{\backprime\backprime}.47.$$

The centesimal or decimal method was proposed by the French mathematicians in the beginning of the present century. But although it possesses many advantages over the established method, they were not considered sufficient to counterbalance the enormous labor which would have been necessary to rearrange all the mathematical tables, books of reference, and records of observations, which would have to be transferred into the decimal system before its advantages could be felt. Thus, the centesimal method has never been used even in France, and in all probability never will be used in practical work.

8. The Circular Measure. — The *unit of circular measure* is the angle subtended at the centre of a circle by an arc equal in length to the radius.

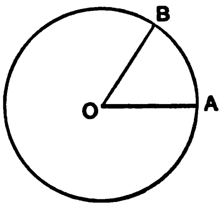
This unit of circular measure is called a *radian*.



Let O be the centre of a circle whose radius is r.

Let the arc AB be equal to the radius $OA = r$.

Then, since angles at the centre of a circle are in the same ratio as their intercepted arcs (Geom., Art. 234), and since the ratio of the circumference of a circle to its diameter is $\pi = 3.14159265$ (Geom., Art. 436),

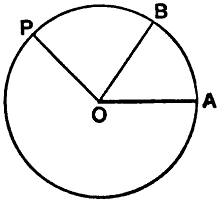


$$\therefore \text{ angle AOB} : 4 \text{ rt. angles} :: \text{arc AB} : \text{circumference},$$
$$:: r : 2\pi r :: 1 : 2\pi.$$

$$\therefore \text{ angle AOB} = \frac{4 \text{ rt. angles}}{2\pi} = \frac{2 \text{ rt. angles}}{\pi}.$$

$$\therefore \text{ a radian} = \text{angle AOB} = \frac{180^\circ}{3.14159265} = 57^\circ.2957795$$
$$= 3437'.74677 = 206264''.806.$$

Therefore, *the radian is the same for all circles*, and $= 57^\circ.2957795$.



Let ABP be any circle; let the angle AOB be the radian; and let AOP be any other angle.

Then arc AB = radius OA.

$$\therefore \text{ angle AOP} : \text{angle AOB}$$
$$:: \text{arc AP} : \text{arc AB};$$

or $\quad \text{angle AOP} : \text{radian} :: \text{arc AP} : \text{radius}.$

$$\therefore \text{ angle AOP} = \frac{\text{arc AP}}{\text{radius}} \times \text{radian}.$$

The *measure* of any quantity is the number of times it contains the *unit* of measure (Art. 2).

$$\therefore \text{ the circular measure of angle AOP} = \frac{\text{arc AP}}{\text{radius}}.$$

NOTE 1. — The student will notice that a radian is a little less than an angle of an equilateral triangle, *i.e.*, of 60°.

Angles expressed in circular measure are usually denoted by Greek letters, $\alpha, \beta, \gamma, \ldots, \phi, \theta, \psi, \ldots$.

The circular measure is employed in the various branches of Analytical Mathematics, in which the angle under consideration is almost always expressed by a *letter*.

NOTE 2. — The student cannot too carefully notice that unless an *angle* is obviously referred to, the letters $\alpha, \beta, \ldots, \theta, \phi, \ldots$ stand for *mere numbers*. Thus, π *stands for a number, and a number only, viz.*, 3.14159 ..., but in the expression '*the angle* π,' that is, 'the angle 3.14159 ...,' there must be *some unit* understood. The *unit understood* here is a radian, and therefore 'the angle π' stands for 'π radians' or 3.14159 ... radians, that is, *two right angles.*

Hence, *when an angle is referred to,* π *is a very convenient abbreviation for two right angles.*

So also 'the angle α or θ' means 'α radians or θ radians.'

The units in the three systems, when expressed in terms of one common standard, two right angles, stand thus :

The unit in the Sexagesimal Method $= \dfrac{1}{180}$ of 2 right angles.

" " " " Centesimal " $= \dfrac{1}{200}$ " " " "

" " " " Circular " $= \dfrac{1}{\pi}$ " " " "

If D, G, and θ denote the number of degrees, grades, and radians respectively in any angle, then

$$\frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi}, \qquad (1)$$

because each fraction is the ratio of the angle to two right angles.

9. Comparison of the Sexagesimal and Centesimal Measures of an Angle. — Although the centesimal method was never in general use among mathematicians, and is now totally abandoned everywhere, yet it still possesses some interest, as it shows the application of the decimal system to the measurement of angles.

From (1) of Art. 8 we have

$$\frac{D}{180} = \frac{G}{200}.$$

$$\therefore\ D = \frac{9}{10}G, \text{ and } G = \frac{10}{9}D.$$

EXAMPLES.

1. Express 49° 15′ 35″ in centesimal measure.

First express the angle in degrees and decimals of a degree thus:

$$\begin{array}{r} 60\underline{)\ 35''\qquad} \\ 60\underline{)\ 15'.58\dot{3}\quad} \\ 49^{\circ}.2597\dot{2} \\ \underline{\qquad\qquad 10} \\ 9\underline{)\ 492.597\dot{2}\quad} \\ 54^{g}.733024\cdots. \end{array}$$

$$\therefore\ 49^{\circ}\ 15'\ 35'' = 54^{g}\ 73^{\backprime}\ 30^{\backprime\backprime}.24\cdots.$$

2. Express $87^{g}\ 2^{`}\ 25^{``}$ in degrees, etc.

First express the angle in grades and decimals of a grade thus:

$$\begin{array}{r} 87^{g}\ 2^{`}\ 25^{``} = 87^{g}.0225 \\ .9 \\ \hline 78.32025 \\ 60 \\ \hline 19.215 \\ 60 \\ \hline 12.9 \end{array}$$

$$\therefore\ 87^{g}\ 2^{`}\ 25^{``} = 78^{\circ}\ 19'\ 12''.9.$$

Find the number of grades, minutes, and seconds in the following angles:

3. $51^{\circ}\ 4'\ 30''$. *Ans.* $56^{g}\ 75^{`}\ 0^{``}$.
4. $45^{\circ}\ 33'\ 3''$. $50^{g}\ 61^{`}\ 20^{``}.37$.
5. $27^{\circ}\ 15'\ 46''$. $30^{g}\ 29^{`}\ 19^{``}.75\cdots$.
6. $157^{\circ}\ 4'\ 9''$. $174^{g}\ 52^{`}\ 12^{``}.962\cdots$.

Find the number of degrees, minutes, and seconds in the following angles:

7. $19^{g}\ 45^{`}\ 95^{``}$. *Ans.* $17^{\circ}\ 30'\ 48''.78$.
8. $124^{g}\ 5^{`}\ 8^{``}$. $111^{\circ}\ 38'\ 44''.592$.
9. $55^{g}\ 18^{`}\ 35^{``}$. $49^{\circ}\ 39'\ 54''.54$.

10. Comparison of the Sexagesimal and Circular Measures of an Angle.

From (1) of Art. 8 we have

$$\frac{D}{180} = \frac{\theta}{\pi}.$$

$$\therefore\ D = \frac{180\,\theta}{\pi},\ \text{and}\ \theta = \frac{D\pi}{180}.$$

EXAMPLES.

1. Find the number of degrees in the angle whose circular measure is $\frac{1}{2}$.

Here
$$\theta = \frac{1}{2}.$$
$$\therefore\ D = \frac{180}{\pi} \times \frac{1}{2} = \frac{90}{\pi}$$
$$= \frac{90 \times 7}{22} = 28^\circ\ 38'\ 10''\tfrac{10}{11},$$
where $\frac{22}{7}$ is used for π.

2. Find the circular measure of the angle 59° 52′ 30″.

Express the angle in degrees and decimals of a degree thus:
$$\begin{array}{r} 60\underline{)52.5}. \\ 59.875 \end{array}$$
$$\therefore\ \theta = \tfrac{59.875}{180}\pi = (.333 \cdots)\,\pi = 1.0453 \cdots.$$

3. Express, in degrees, the angles whose circular measures are $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$, $\frac{2}{3}\pi$.

Note 1. — The student should especially accustom himself to express readily in circular measure an angle which is given in degrees.

4. Express in circular measure the following angles:

60°, 22° 30′, 11° 15′, 270°. *Ans.* $\frac{\pi}{3}$, $\frac{\pi}{8}$, $\frac{\pi}{16}$, $\frac{3\pi}{2}$.

5. Express in circular measure 3° 12′, and find to seconds the angle whose circular measure is .8.

$\left(\text{Take } \pi = \frac{22}{7}.\right)$ *Ans.* $\frac{4\pi}{225}$, $45^\circ\ 49'\ 5''\tfrac{5}{11}$.

6. One angle of a triangle is 45°, and the circular measure of another is 1.5. Find the third angle in degrees.

Ans. $49^\circ\ 5'\ 27''\tfrac{3}{11}$.

Note 2. — Questions in which angles are expressed in different systems of measurement are easily solved by expressing each angle in right angles.

7. The sum of the measure of an angle in degrees and twice its measure in radians is $23\frac{2}{7}$; find its measure in degrees ($\pi = \frac{22}{7}$).

Let the angle contain x right angles.

Then the measure of the angle in degrees $= 90x$.

" " " " " " " radians $= \frac{\pi}{2}x$.

$$\therefore\ 90x + \pi x = 23\tfrac{2}{7};$$

$$\therefore\ 90x + \tfrac{22}{7}x = \frac{163}{7};$$

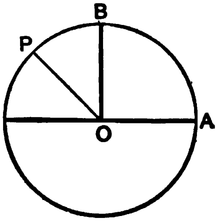
$$\therefore\ 652x = 163;\ \therefore\ x = \frac{1}{4}.$$

$\therefore$ the angle is $\frac{1}{4}$ of $90° = 22\frac{1}{2}°$.

8. The difference between two angles is $\frac{\pi}{9}$, and their sum is $56°$; find the angles in degrees. *Ans.* $38°$, $18°$.

11. General Measure of an Angle. — In Euclidian geometry and in practical applications of trigonometry, angles are generally considered to be less than two right angles; but in the theoretical parts of mathematics, angles are treated as quantities which may be of any magnitude whatever.

Thus, when we are told that an angle is in some particular quadrant, say the second (Art. 5), we know that the position in which the revolving line *stops* is in the second quadrant. But there is an unlimited number of *angles* having the same final position, OP.



The revolving line OP may pass from OA to OP, not only by describing the arc ABP, but by moving through a whole revolution *plus* the arc ABP, or through any number of revolutions *plus* the arc ABP.

For example, the final position of OP may represent *geometrically* all the following angles:

Angle AOP $= 130°$, or $360° + 130°$, or $720° + 130°$, or $-360° + 130°$, or $-720° + 130°$, etc.

Let A be an angle between 0 and 90°, and let n be any *whole* number, positive or negative. Then

(1) $2n \times 180° + A$ represents algebraically an angle in the *first* quadrant.
(2) $2n \times 180° - A$ represents algebraically an angle in the *fourth* quadrant.
(3) $(2n + 1)\ 180° - A$ represents algebraically an angle in the *second* quadrant.
(4) $(2n + 1)\ 180° + A$ represents algebraically an angle in the *third* quadrant.

In circular measure the corresponding expressions are

(1) $2n\pi + \theta$, (2) $2n\pi - \theta$, (3) $(2n+1)\pi - \theta$, (4) $(2n+1)\pi + \theta$.

EXAMPLES.

State in which quadrant the revolving line will be after describing the following angles:

(1) $120°$, (2) $340°$, (3) $490°$, (4) $-100°$,
(5) $-380°$, (6) $\frac{2}{3}\pi$, (7) $10\pi + \frac{\pi}{4}$.

12. Complement and Supplement of an Angle or Arc. — The *complement* of an angle or arc is the remainder obtained by subtracting it from a right angle or 90°.

The *supplement* of an angle or arc is the remainder obtained by subtracting it from *two right angles* or 180°.

Thus, the complement of A is $(90° - A)$.
The complement of 190° is $(90° - 190°) = -100°$.
The supplement of A is $(180° - A)$.
The supplement of 200° is $(180° - 200°) = -20°$.
The complement of $\frac{5}{4}\pi$ is $\left(\frac{\pi}{2} - \frac{5}{4}\pi\right) = -\frac{3}{4}\pi$.
The supplement of $\frac{3}{4}\pi$ is $(\pi - \frac{3}{4}\pi) = \frac{1}{4}\pi$.

EXAMPLES.

1. If 192 square inches be represented by the number 12, what is the unit of linear measurement? *Ans.* 4 inches.

2. If 1000 square inches be represented by the number 40, what is the unit of linear measurement? *Ans.* 5 inches.

3. If 2000 cubic inches be represented by the number 16, what is the unit of linear measurement? *Ans.* 5 inches.

4. The length of an Atlantic cable is 2300 miles and the length of the cable from England to France is 21 miles. Express the length of the first in terms of the second as unit. *Ans.* $109\frac{11}{21}$.

5. Find the measure of a miles when b yards is the unit. *Ans.* $\frac{1760a}{b}$.

6. The ratio of the area of one field to that of another is 20 : 1, and the area of the first is half a square mile. Find the number of square yards in the second. *Ans.* 77440.

7. A certain weight is 3.125 tons. What is its measure in terms of 4 cwt.? *Ans.* 15.625.

Express the following 12 angles in centesimal measure:

8. $42^\circ\ 15'\ 18''$. *Ans.* $46^{g}\ 95^{\backprime}$.
9. $63^\circ\ 19'\ 17''$. $70^{g}\ 35^{\backprime}\ 70^{\backprime\backprime}.98\cdots$.
10. $103^\circ\ 15'\ 45''$. $114^{g}\ 73^{\backprime}\ 61^{\backprime\backprime}.1$.
11. $19^\circ\ 0'\ 18''$. $21^{g}\ 11^{\backprime}\ 66^{\backprime\backprime}.\dot{6}$.
12. $143^\circ\ 9'\ 0''$. $159^{g}\ 5^{\backprime}\ 55^{\backprime\backprime}.\dot{5}$.
13. $300^\circ\ 15'\ 58''$. $333^{g}\ 62^{\backprime}\ 90^{\backprime\backprime}.\dot{1}23456789\dot{0}$.
14. $27^\circ\ 41'\ 51''$. $30^{g}.775$.
15. $67^\circ.4325$. $74^{g}.925$.
16. $8^\circ\ 15'\ 27''$. $9^{g}\ 17^{\backprime}\ 50^{\backprime\backprime}$.
17. $97^\circ\ 5'\ 15''$. $107^{g}\ 87^{\backprime}\ 50^{\backprime\backprime}$.
18. $16^\circ\ 14'\ 19''$. $18^{g}\ 4^{\backprime}\ 29^{\backprime\backprime}\cdots$.
19. $132^\circ\ 6'$. $146^{g}\ 77^{\backprime}\ 77^{\backprime\backprime}.\dot{7}$.

Express the following 11 angles in degrees, minutes, and seconds:

20. $105^{g}\,52^{\backprime}\,75^{\backprime\backprime}$. *Ans.* $94^\circ\,58'\,29''.1$.

21. $82^{g}\,9^{\backprime}\,54^{\backprime\backprime}$. $73^\circ\,53'\,9''.096$.

22. $70^{g}\,15^{\backprime}\,92^{\backprime\backprime}$. $63^\circ\,8'\,35''.808$.

23. $15^{g}\,0^{\backprime}\,15^{\backprime\backprime}$. $13^\circ\,30'\,4''.86$.

24. $154^{g}\,7^{\backprime}\,24^{\backprime\backprime}$. $138^\circ\,39'\,54''.576$.

25. $324^{g}\,13^{\backprime}\,88^{\backprime\backprime}.7$. $291^\circ\,43'\,29''.9388$.

26. $10^{g}\,42^{\backprime}\,50^{\backprime\backprime}$. $9^\circ\,22'\,57''$.

27. $20^{g}\,77^{\backprime}\,50^{\backprime\backprime}$. $18^\circ\,41'\,51''$.

28. $8^{g}\,75^{\backprime}$. $7^\circ\,52'\,30''$.

29. $170^{g}\,45^{\backprime}\,35^{\backprime\backprime}$. $153^\circ\,24'\,29''.34$.

30. $24^{g}\,0^{\backprime}\,25^{\backprime\backprime}$. $21^\circ\,36'\,8''.1$.

Express in circular measure the following angles:

31. 315°, $24^\circ\,13'$. *Ans.* $\frac{7}{4}\pi$, $\dfrac{1453\pi}{10800}$.

32. $95^\circ\,20'$, $12^\circ\,5'\,4''$. $\dfrac{143\pi}{270}$, $\dfrac{2719\pi}{40500}$.

33. $22\frac{1}{2}^\circ$, 1°, $57^\circ.295$. $\dfrac{\pi}{8}$, $\dfrac{\pi}{180}$, 1 radian.

34. 120°, 45°, 270°. 2.09439, $\dfrac{\pi}{4}$, $\frac{3}{2}\pi$.

35. 360°, $3\frac{1}{2}$ rt. angles. 2π, $\frac{7}{4}\pi$.

Express in degrees, etc., the angles whose circular measures are:

36. $\frac{5}{8}\pi$, $\frac{2}{3}\pi$, $\dfrac{1}{2}$. *Ans.* $112^\circ.5$, 120°, $\dfrac{90}{\pi}$ degrees.

37. $\dfrac{1}{4}$, $\dfrac{1}{6}$, $\dfrac{2}{3}$. $\dfrac{45}{\pi}$ degrees, $\dfrac{30}{\pi}$ degrees, $\dfrac{120}{\pi}$ degrees.

38. $\dfrac{5}{6}$, $.7854$. $47^\circ\,43'\,38''\frac{2}{11}$, 45°.

39. $4\frac{1}{2}$, $\frac{1}{12}\pi$, 2.504. *Ans.* 257° 49′ 43″.39, 15°, 143°.468.

40. .0234, 1.234, $\frac{2}{3}$. 1°20′27″, 70°42′11″, 38°11′50″.

41. Find the number of radians in an angle at the centre of a circle of radius 25 feet, which intercepts an arc of $37\frac{1}{2}$ feet. *Ans.* $1\frac{1}{2}$.

42. Find the number of degrees in an angle at the centre of a circle of radius 10 feet, which intercepts an arc of 5π feet. *Ans.* 90°.

43. Find the number of right angles in an angle at the centre of a circle of radius $3\frac{2}{11}$ inches, which intercepts an arc of 2 feet. *Ans.* $4\frac{4}{5}$.

44. Find the length of the arc subtending an angle of $4\frac{1}{2}$ radians at the centre of a circle whose radius is 25 feet. *Ans.* $112\frac{1}{2}$ ft.

45. Find the length of an arc of 80° on a circle of 4 feet radius. *Ans.* $5\frac{37}{63}$ ft.

46. The angle subtended by the diameter of the Sun at the eye of an observer is 32′: find approximately the diameter of the Sun if its distance from the observer be 90 000 000 miles. *Ans.* 838 000 miles.

47. A railway train is travelling on a curve of half a mile radius at the rate of 20 miles an hour: through what angle has it turned in 10 seconds? *Ans.* $6\frac{4}{11}$ degrees.

48. If the radius of a circle be 4000 miles, find the length of an arc which subtends an angle of 1″ at the centre of the circle. *Ans.* About 34 yards.

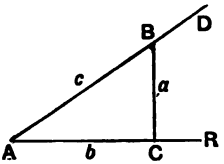
49. On a circle of 80 feet radius it was found that an angle of 22° 30′ at the centre was subtended by an arc 31 ft. 5 in. in length: hence calculate to four decimal places the numerical value of the ratio of the circumference of a circle to its diameter. *Ans.* $3.141\dot{6}$.

50. Find the number of radians in 10″ correct to four significant figures (use $\frac{355}{113}$ for π). *Ans.* .00004848.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

13. Definitions of the Trigonometric Functions. — Let RAD be an angle; in AD, one of the lines containing the angle, take any point B, and from B draw BC perpendicular to the other line AR, thus forming a right triangle ABC, right-angled at C. Then denoting the angles by the capital letters A, B, C, respectively, and the three sides opposite these angles by the corresponding small italics, a, b, c,* we have the following definitions:



$$\frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}} \text{ is called the } \textit{sine} \text{ of the angle A.}$$

$$\frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}} \text{ is called the } \textit{cosine} \text{ of the angle A.}$$

$$\frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}} \text{ is called the } \textit{tangent} \text{ of the angle A.}$$

$$\frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}} \text{ is called the } \textit{cotangent} \text{ of the angle A.}$$

$$\frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}} \text{ is called the } \textit{secant} \text{ of the angle A.}$$

$$\frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}} \text{ is called the } \textit{cosecant} \text{ of the angle A.}$$

If the cosine of A be subtracted from unity, the remainder is called the *versed sine* of A. If the sine of A be subtracted from unity, the remainder is called the *coversed sine* of A; the latter term is hardly ever used in practice.

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* The letters *a, b, c* are *numbers*, being the number of times the lengths of the sides contain some chosen unit of length.

The words sine, cosine, etc., are abbreviated, and the functions of an angle A are written thus: sin A, cos A, tan A, cot A, sec A, cosec A, vers A, covers A.

The following is the verbal enunciation of these definitions:

The sine of an angle is the ratio of the opposite side to the hypotenuse; or $\sin A = \frac{a}{c}$.

The cosine of an angle is the ratio of the adjacent side to the hypotenuse; or $\cos A = \frac{b}{c}$.

The tangent of an angle is the ratio of the opposite side to the adjacent side; or $\tan A = \frac{a}{b}$.

The cotangent of an angle is the ratio of the adjacent side to the opposite side; or $\cot A = \frac{b}{a}$.

The secant of an angle is the ratio of the hypotenuse to the adjacent side; or $\sec A = \frac{c}{b}$.

The cosecant of an angle is the ratio of the hypotenuse to the opposite side; or $\operatorname{cosec} A = \frac{c}{a}$.

The versed sine of an angle is unity minus the cosine of the angle; or $\operatorname{vers} A = 1 - \cos A = 1 - \frac{b}{c}$.

The coversed sine of an angle is unity minus the sine of the angle; or $\operatorname{covers} A = 1 - \sin A = 1 - \frac{a}{c}$.

These ratios are called *Trigonometric Functions.* The student should carefully commit them to memory, as upon them is founded the whole theory of Trigonometry.

These functions are, it will be observed, not *lengths,* but *ratios* of one length to another; that is, they are ***abstract numbers***, simply numerical quantities; and they remain unchanged so long as the angle remains unchanged, as will be proved in Art. 14.

It is clear from the above definitions that

$$\operatorname{cosec} A = \frac{1}{\sin A}, \text{ or } \sin A = \frac{1}{\operatorname{cosec} A},$$
$$\sec A = \frac{1}{\cos A}, \text{ or } \cos A = \frac{1}{\sec A},$$
$$\tan A = \frac{1}{\cot A}, \text{ or } \cot A = \frac{1}{\tan A}.$$

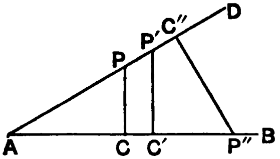
The powers of the Trigonometric functions are expressed as follows:

$$(\sin A)^2 \text{ is written } \sin^2 A,$$
$$(\cos A)^3 \text{ is written } \cos^3 A,$$

and so on.

Note.—The student must notice that 'sin A' is a *single symbol*, the *name* of a *number*, or *fraction* belonging to the angle A. Also $\sin^2 A$ is an abbreviation for $(\sin A)^2$, *i.e.*, for $(\sin A) \times (\sin A)$. Such abbreviations are used for *convenience*.

14. The Trigonometric Functions are always the Same for the Same Angle.—Let BAD be any angle; in AD take P, P′, any two points, and draw PC, P′C′ perpendicular to AB. Take P″, any point in AB, and draw P″C″ perpendicular to AD.



Then the three triangles PAC, P′AC′, P″AC″ are equiangular, since they are right-angled, and have a common angle at A: therefore they are similar.

$$\therefore \frac{PC}{AP} = \frac{P'C'}{AP'} = \frac{P''C''}{AP''}.$$

But each of these ratios is the sine of the angle A. Thus, sin A is the same *whatever* be the position of the point P on *either* of the lines containing the angle A.

Therefore $\sin A$ is always the same. A similar proof may be given for each of the other functions.

In the right triangle of Art. 13, show that

$$a = c \sin A = c \cos B = b \tan A = b \cot B,$$
$$b = a \cot A = a \tan B = c \cos A = c \sin B,$$
$$c = a \operatorname{cosec} A = a \sec B = b \sec A = b \operatorname{cosec} B.$$

Note. — These results should be carefully noticed, as they are of frequent use in the solution of right triangles and elsewhere.

EXAMPLES.

1. Calculate the value of the functions, sine, cosine, etc., of the angle A in the right triangles whose sides a, b, c are respectively (1) 8, 15, 17; (2) 40, 9, 41; (3) 196, 315, 371; (4) 480, 31, 481; (5) 1700, 945, 1945.

Ans. (1) $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$, etc.;
(2) $\sin A = \frac{40}{41}$, $\cos A = \frac{9}{41}$, etc.;
(3) $\sin A = \frac{28}{53}$, $\tan A = \frac{28}{45}$, etc.;
(4) $\sin A = \frac{480}{481}$, $\tan A = \frac{480}{31}$, etc.;
(5) $\sin A = \frac{340}{389}$, $\tan A = \frac{340}{189}$, etc.

In a right triangle, given:

2. $a = \sqrt{m^2 + n^2}$, $b = \sqrt{2mn}$; calculate $\sin A$. *Ans.* $\frac{\sqrt{m^2 + n^2}}{m + n}$.

3. $a = \sqrt{m^2 - 2mn}$, $b = n$; calculate $\sec A$. $\frac{m - n}{n}$.

4. $a = \sqrt{m^2 + mn}$, $c = m + n$; calculate $\tan A$. $\sqrt{\frac{m}{n}}$.

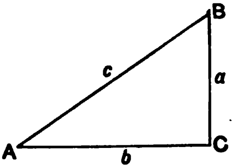
5. $a = 2mn$, $b = m^2 - n^2$; calculate $\cos A$. $\frac{m^2 - n^2}{m^2 + n^2}$.

6. $\sin A = \frac{3}{5}$, $c = 200.5$; calculate a. 120.3.

7. $\cos A = .44$, $c = 30.5$; calculate b. 13.42.

8. $\tan A = \frac{11}{3}$, $b = \frac{27}{11}$; calculate c. $\frac{9}{11}\sqrt{130}$.

15. Functions of Complemental Angles. — In the rt. $\triangle ABC$ we have



$$\sin A = \frac{a}{c}, \text{ and } \cos B = \frac{a}{c}. \quad \text{(Art. 13.)}$$

$$\therefore \sin A = \cos B.$$

But B is the complement of A, since their sum is a right angle, or 90°; *i.e.*, $B = 90° - A$.

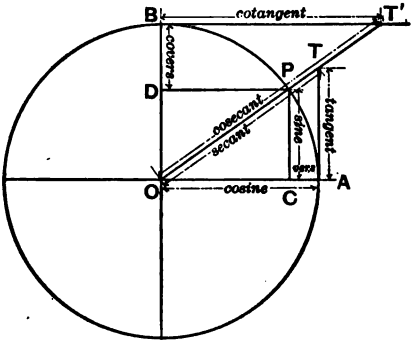
$$\begin{aligned}
&\therefore & \sin A &= \cos B &&= \cos(90° - A) &&= \frac{a}{c}.\\
&\text{Also} & \cos A &= \sin B &&= \sin(90° - A) &&= \frac{b}{c},\\
&& \tan A &= \cot B &&= \cot(90° - A) &&= \frac{a}{b},\\
&& \cot A &= \tan B &&= \tan(90° - A) &&= \frac{b}{a},\\
&& \sec A &= \operatorname{cosec} B &&= \operatorname{cosec}(90° - A) &&= \frac{c}{b},\\
&& \operatorname{cosec} A &= \sec B &&= \sec(90° - A) &&= \frac{c}{a},\\
&& \operatorname{vers} A &= \operatorname{covers} B &&= \operatorname{covers}(90° - A) &&= 1 - \frac{b}{c},\\
&& \operatorname{covers} A &= \operatorname{vers} B &&= \operatorname{vers}(90° - A) &&= 1 - \frac{a}{c}.
\end{aligned}$$

Therefore the *sine, tangent, secant, and versed sine of an angle are equal respectively to the cosine, cotangent, cosecant, and coversed sine of the complement of the angle.*

16. Representation of the Trigonometric Functions by Straight Lines. — The Trigonometric functions were formerly defined as being certain *straight lines* geometrically connected with the *arc* subtending the angle at the centre of a circle of given radius.

Thus, let AP be the arc of a circle subtending the angle AOP at the centre.

Draw the tangents AT, BT′ meeting OP produced to T′, and draw PC, PD ⊥ to OA, OB.



Then	PC	was called the	sine	of the arc AP.
	OC	“	cosine	“
	AT	“	tangent	“
	BT′	“	cotangent	“
	OT	“	secant	“
	OT′	“	cosecant	“
	AC	“	versed sine	“
	BD	“	coversed sine	“

Since any arc is the measure of the angle at the centre which the arc subtends (Art. 5), the above functions of the arc AP are also functions of the angle AOP.

It should be noticed that the old functions of the *arc* above given, when divided by the radius of the circle, become the modern functions of the *angle* which the arc subtends at the centre. If, therefore, the radius be taken as *unity*, the old functions of the arc AP become the modern functions of the angle AOP.

Thus, representing the arc AP, or the angle AOP by θ, we have, when $OA = OP = 1$,

$$\sin\theta = \frac{PC}{OP} = \frac{PC}{1} = PC,$$

$$\tan\theta = \frac{AT}{OA} = \frac{AT}{1} = AT,$$

and similarly for the other functions.

Therefore, in a circle whose radius is unity, the *Trigonometric functions* of an arc, or of the angle at the centre measured by that arc, may be defined as follows:

The sine is the perpendicular let fall from one extremity of the arc upon the diameter passing through the other extremity.

The cosine is the distance from the centre of the circle to the foot of the sine.

The tangent is the line which touches one extremity of the arc and is terminated by the diameter produced passing through the other extremity.

The secant is the portion of the diameter produced through one extremity of the arc which is intercepted between the centre and the tangent at the other extremity.

The versed sine is the part of the diameter intercepted between the beginning of the arc and the foot of the sine.

Since the lines PD or OC, BT′, OT′, and BD are respectively the sine, tangent, secant, and versed sine of the arc BP, which (Art. 12) is the complement of AP, we see that *the cosine, the cotangent, the cosecant, and the coversed sine of an arc are respectively the sine, the tangent, the secant, and the versed sine of its complement.*

EXAMPLES.

1. Prove $\tan A \sin A + \cos A = \sec A$.
2. " $\cot A \cos A + \sin A = \operatorname{cosec} A$.
3. " $(\tan A - \sin A)^2 + (1 - \cos A)^2 = (\sec A - 1)^2$.
4. " $\tan A + \cot A = \sec A \operatorname{cosec} A$.
5. " $(\sin A + \cos A) \div (\sec A + \operatorname{cosec} A) = \sin A \cos A$.
6. " $(1 + \tan A)^2 + (1 + \cot A)^2 = (\sec A + \operatorname{cosec} A)^2$.

7. Given $\tan A = \cot 2A$; find A.

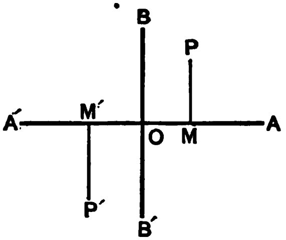
8. " $\sin A = \cos 3A$; find A.

9. " $\sin A = \cos(45° - \frac{1}{2}A)$; find A.

10. " $\tan A = \cot 6A$; find A.

11. " $\cot A = \tan(45° + A)$; find A.

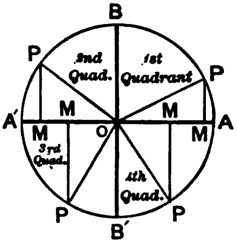
17. Positive and Negative Lines. — Let AA′ and BB′ be two perpendicular right lines intersecting at the point O. Then the position of any point in the line AA′ or BB′ will be determined if we know the distance of the point from O, and if we know also upon which side of O the point lies. It is therefore convenient to employ the algebraic signs + and −, so that if distances measured along the fixed line OA or OB from O in one direction be considered positive, distances measured along OA′ or OB′ in the opposite direction from O will be considered negative.



This *convention*, as it is called, is extended to lines parallel to AA′ and BB′; and it is customary to consider distances measured from BB′ towards the *right* and from AA′ *upwards* as *positive*, and consequently distances measured from BB′ towards the *left* and from AA′ *downwards* as *negative*.

18. Trigonometric Functions of Angles of Any Magnitude. — In the definitions of the trigonometric functions given in Art. 13 we considered only acute angles, *i.e.*, angles in the first quadrant (Art. 5), since the angle was assumed to be one of the acute angles of a right triangle. We shall now show that these definitions apply to angles of any magnitude, and that the functions vary in *sign* according to the *quadrant* in which the angle happens to be.

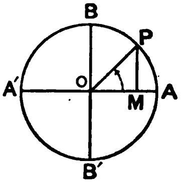
Let AOP be an angle of any magnitude formed by OP revolving from an initial position OA. Draw PM ⊥ to AA′. Consider OP as always positive. Let the angle AOP be denoted by A; then whatever be the magnitude of the angle A, the definitions of the trigonometric functions are



$$\sin A = \frac{MP}{OP}, \ \cos A = \frac{OM}{OP}, \ \tan A = \frac{MP}{OM},$$

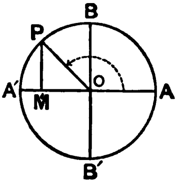
$$\sec A = \frac{OP}{OM}, \ \cot A = \frac{OM}{MP}, \ \operatorname{cosec} A = \frac{OP}{MP}.$$

I. When A lies in the 1st quadrant, MP is *positive* because measured from M *upwards*, OM is *positive* because measured from O towards the *right* (Art. 17), and OP is *positive.*



Hence in the *first* quadrant all the functions are *positive.*

II. When A lies in the 2d quadrant, as the *obtuse* angle AOP, MP is *positive* because measured from M *upwards,* OM is *negative* because measured from O towards the *left* (Art. 17), and OP is *positive.*



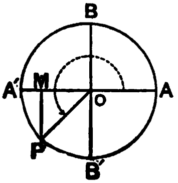
Hence in the *second* quadrant

$$\sin A = \frac{MP}{OP} \text{ is } \textit{positive;}$$

$$\cos A = \frac{OM}{OP} \text{ is } \textit{negative;}$$

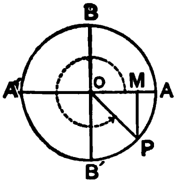
$$\tan A = \frac{MP}{OM} \text{ is } \textit{negative;}$$

and therefore sec A and cot A are *negative,* and cosec A is *positive* (Art. 13).



III. When A lies in the 3d quadrant, as the *reflex* angle AOP, MP is *negative* because measured from M *downwards*, OM is *negative*, and OP is *positive*.

Hence in the *third* quadrant the *sine*, *cosine*, *secant*, and *cosecant*, are *negative*, but the *tangent* and *cotangent* are *positive*.



IV. When A lies in the 4th quadrant, as the reflex angle AOP, MP is *negative*, OM is *positive*, and OP is *positive*.

Hence in the *fourth* quadrant the *sine*, *tangent*, *cotangent*, and *cosecant* are *negative*, but the *cosine* and *secant* are *positive*.

The signs of the different functions are shown in the annexed table.

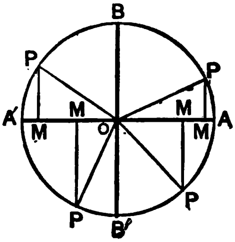
QUADRANT.	I.	II.	III.	IV.
Sin and cosec	+	+	−	−
Cos and sec	+	−	−	+
Tan and cot	+	−	+	−

NOTE. — It is apparent from this table that the signs of all the functions in any quadrant are known when those of the sine and cosine are known. The tangent and cotangent are + or −, according as the sine and cosine have like or different signs.

19. Changes in the Value of the Sine as the Angle increases from 0° to 360°. — Let A denote the angle AOP described by the revolution of OP from its initial position OA through 360°. Then, PM being drawn perpendicular to AA′,

$$\sin A = \frac{MP}{OP},$$

whatever be the magnitude of the angle A.



When the angle A is 0°, P coincides with A, and MP is zero; therefore $\sin 0° = 0$.

As A increases from 0° to 90°, MP increases from zero to OB or OP, and is *positive;* therefore $\sin 90° = 1$.

Hence in the 1st quadrant $\sin A$ is *positive,* and increases from 0 to 1.

As A increases from 90° to 180°, MP decreases from OP to zero, and is *positive;* therefore $\sin 180° = 0$.

Hence in the 2d quadrant $\sin A$ is *positive,* and decreases from 1 to 0.

As A increases from 180° to 270°, MP increases from zero to OP, and is *negative;* therefore $\sin 270° = -1$.

Hence in the 3d quadrant $\sin A$ is *negative,* and decreases algebraically from 0 to -1.

As A increases from 270° to 360°, MP decreases from OP to zero, and is *negative;* therefore $\sin 360° = 0$.

Hence in the 4th quadrant $\sin A$ is *negative,* and increases algebraically from -1 to 0.

20. Changes in the Cosine as the Angle increases from 0° to 360°. — In the figure of Art. 19

$$\cos A = \frac{OM}{OP}.$$

When the angle A is 0°, P coincides with A, and OM = OP; therefore $\cos 0° = 1$.

As A increases from 0° to 90°, OM decreases from OP to zero and is *positive;* therefore $\cos 90° = 0$.

Hence in the 1st quadrant $\cos A$ is *positive,* and decreases from 1 to 0.

As A increases from 90° to 180°, OM increases from zero to OP, and is *negative;* therefore $\cos 180° = -1$.

Hence in the 2d quadrant $\cos A$ is *negative,* and decreases algebraically from 0 to -1.

As A increases from 180° to 270°, OM decreases from OP to zero, and is *negative;* therefore $\cos 270° = 0$.

Hence in the 3d quadrant cos A is *negative*, and increases algebraically from -1 to 0.

As A increases from 270° to 360°, OM increases from zero to OP, and is *positive;* therefore $\cos 360° = 1$.

Hence in the 4th quadrant cos A is *positive*, and increases from 0 to 1.

21. Changes in the Tangent as the Angle increases from 0° to 360°.—In the figure of Art. 19

$$\tan A = \frac{MP}{OM}.$$

When A is 0°, MP is zero, and $OM = OP$; therefore $\tan 0° = 0$.

As A increases from 0° to 90°, MP increases from zero to OP, and OM decreases from OP to zero, so that on both accounts tan A increases numerically; therefore $\tan 90° = \infty$.

Hence in the 1st quadrant tan A is *positive*, and increases from 0 to ∞.

As A increases from 90° to 180°, MP decreases from OP to zero, and is *positive*, OM becomes *negative* and decreases algebraically from zero to -1; therefore $\tan 180° = 0$.

Hence in the 2d quadrant tan A is *negative*, and increases algebraically from $-\infty$ to 0.

When A passes into the 2d quadrant, and is only just greater than 90°, tan A changes from $+\infty$ to $-\infty$.

As A increases from 180° to 270°, MP increases from zero to OP, and is *negative*, OM decreases from OP to zero, and is *negative;* therefore $\tan 270° = \infty$.

Hence in the 3d quadrant tan A is *positive*, and increases from 0 to ∞.

As A increases from 270° to 360°, MP decreases from OP to zero, and is *negative*, OM increases from zero to OP, and is *positive;* therefore $\tan 360° = 0$.

Hence in the 4th quadrant tan A is *negative*, and increases algebraically from $-\infty$ to 0.

The student is recommended to trace in a manner similar to the above the changes in the other functions, *i.e.*, the cotangent, secant, and cosecant, and to see that his results agree with those given in the following table.

22. Table giving the Changes of the Trigonometric Functions in the Four Quadrants.

QUADRANT.	I.	II.	III.	IV.
sin varies from	+ 0 to 1	+ 1 to 0	− 0 to −1	− −1 to 0
cos “ “	+ 1 to 0	− 0 to −1	− −1 to 0	+ 0 to 1
tan “ “	+ 0 to ∞	− −∞ to 0	+ 0 to ∞	− −∞ to 0
cot “ “	+ ∞ to 0	− 0 to ∞	+ ∞ to 0	− 0 to −∞
sec “ “	+ 1 to ∞	− −∞ to −1	− −1 to −∞	+ ∞ to 1
cosec “ “	+ ∞ to 1	+ 1 to ∞	− −∞ to −1	− −1 to −∞
vers “ “	+ 0 to 1	+ 1 to 2	+ 2 to 1	+ 1 to 0

NOTE 1. — The *cosecant, secant,* and *cotangent* of an angle A have the same *sign* as the *sine, cosine,* and *tangent* of A respectively.

The *sine* and *cosine* vary from 1 to −1, passing through the value 0. They are *never greater than unity.*

The *secant* and *cosecant* vary from 1 to −1, passing through the value ∞. They are *never numerically less than unity.*

The *tangent* and *cotangent* are unlimited in value. They have all values from −∞ to +∞.

The *versed sine* and *coversed sine* vary from 0 to 2, and are always positive.

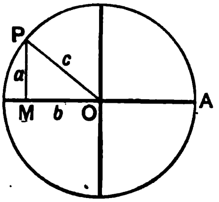
The trigonometric functions change sign in passing through the values 0 and ∞, and through no other values.

In the 1st quadrant the *functions* increase, and the *cofunctions* decrease.

NOTE 2. — From the results given in the above table, it will be seen that, if the value of a trigonometric function be given, we cannot fix on one angle to which it belongs exclusively.

Thus, if the given value of sin A be ½, we know since sin A passes through all values from 0 to 1 as A increases from 0° to 90°, that *one* value of A lies between 0°

and 90°. But since we also know that the value of sin A passes through all values between 1 and 0 as A increases from 90° to 180°, it is evident that there is *another* value of A between 90° and 180° for which sin A = ½.



23. Relations between the Trigonometric Functions of the Same Angle. — Let the radius start from the initial position OA, and *revolve* in either direction, to the position OP. Let θ denote the angle traced out, and let the lengths of the sides PM, MO, OP be denoted by the letters a, b, c.*

The following relations are evident from the definitions (Art. 13):

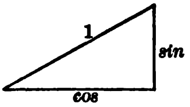
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\quad \cot\theta = \frac{1}{\tan\theta}.$$

I. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}.$

For $\tan\theta = \dfrac{a}{b} = \dfrac{\frac{a}{c}}{\frac{b}{c}} = \dfrac{\sin\theta}{\cos\theta}.$

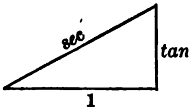
II. $\sin^2\theta + \cos^2\theta = 1.$

For $\sin^2\theta + \cos^2\theta = \dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} = \dfrac{a^2+b^2}{c^2} = 1.$



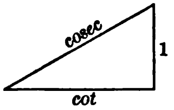
III. $\sec^2\theta = 1 + \tan^2\theta.$

For $\sec^2\theta = \dfrac{c^2}{b^2} = \dfrac{b^2+a^2}{b^2} = 1 + \dfrac{a^2}{b^2} = 1 + \tan^2\theta.$



IV. $\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$

For $\operatorname{cosec}^2\theta = \dfrac{c^2}{a^2} = \dfrac{a^2+b^2}{a^2} = 1 + \dfrac{b^2}{a^2} = 1 + \cot^2\theta.$



Formulæ I., II., III., IV. are very important, and must be remembered.

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* *a, b, c are numbers*, being the number of times the lengths of the sides contain some chosen unit of length.

24. Use of the Preceding Formulæ.

I. *To express all the other functions in terms of the sine.*

Since $\sin^2\theta + \cos^2\theta = 1, \quad \therefore\ \cos\theta = \pm\sqrt{1-\sin^2\theta}.$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$$

$$\cot\theta = \frac{1}{\tan\theta} = \pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}.$$

$$\sec\theta = \frac{1}{\cos\theta} = \pm\frac{1}{\sqrt{1-\sin^2\theta}}.$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

II. *To express all the other functions in terms of the tangent.*

Since $\tan\theta = \frac{\sin\theta}{\cos\theta},$

$$\sin\theta = \tan\theta\cos\theta = \frac{\tan\theta}{\sec\theta} = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}}.$$

$$\cot\theta = \frac{1}{\tan\theta}. \quad \sec\theta = \pm\sqrt{1+\tan^2\theta}.$$

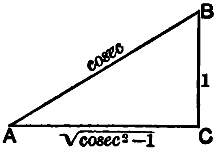
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}.$$

Similarly, any one of the functions of an angle may be expressed in terms of any other function of that angle. The sign of the radical will in all cases depend upon the quadrant in which the angle θ lies.

25. Graphic Method of finding All the Functions in Terms of One of them.

To express all the other functions in terms of the cosecant.

Construct a right triangle ABC, having the side $BC = 1$. Then



$$\operatorname{cosec} A = \frac{AB}{BC} = \frac{AB}{1} = AB.$$

$$\therefore\ AC = \pm\sqrt{\operatorname{cosec}^2 A - 1}.$$

Now
$$\sin A = \frac{BC}{AB} = \frac{1}{\operatorname{cosec} A},$$

$$\cos A = \frac{AC}{AB} = \pm\frac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A},$$

$$\tan A = \frac{BC}{AC} = \pm\frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}},$$

and similarly the other functions may be expressed in terms of cosec A.

26. To find the Trigonometric Functions of 45°. — Let ABC be an isosceles right triangle in which

$$CA = CB.$$

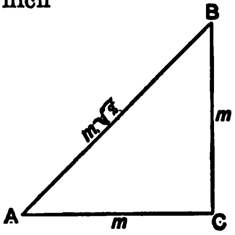
Then $CAB = CBA = 45°.$

Let $AC = m = CB.$

Then

$$\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 = m^2 + m^2 = 2m^2.$$

$$\therefore\ AB = m\sqrt{2}.$$



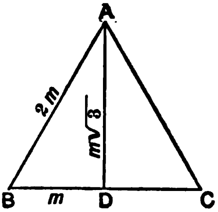
$$\therefore\ \sin 45° = \frac{BC}{AB} = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\cos 45° = \frac{AC}{AB} = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\tan 45° = \frac{BC}{AC} = \frac{m}{m} = 1. \quad \cot 45° = 1.$$

$$\sec 45° = \sqrt{2}. \quad \operatorname{cosec} 45° = \sqrt{2}.$$

27. To find the Trigonometric Functions of 60° and 30°. — Let ABC be an equilateral triangle. Draw AD perpendicular to BC. Then AD bisects the angle BAC and the side BC. Therefore BAD = 30°, and ABD = 60°.



Let $\quad BA = 2m. \quad \therefore \; BD = m.$

Then $\quad AD = \sqrt{4m^2 - m^2} = m\sqrt{3}.$

$$\therefore \sin 60^\circ = \frac{AD}{AB} = \frac{m\sqrt{3}}{2m} = \tfrac{1}{2}\sqrt{3}. \quad \therefore \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$$

$$\cos 60^\circ = \frac{BD}{BA} = \frac{1}{2}. \quad \therefore \sec 60^\circ = 2.$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{m\sqrt{3}}{m} = \sqrt{3}. \quad \therefore \cot 60^\circ = \frac{1}{\sqrt{3}}.$$

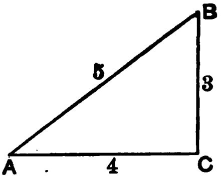
$$\operatorname{vers} 60^\circ = 1 - \cos 60^\circ = 1 - \frac{1}{2} = \frac{1}{2}.$$

Also $\quad \sin 30^\circ = \frac{BD}{AB} = \frac{m}{2m} = \frac{1}{2}. \quad \therefore \operatorname{cosec} 30^\circ = 2.$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{m\sqrt{3}}{2m} = \tfrac{1}{2}\sqrt{3}. \quad \therefore \sec 30^\circ = \frac{2}{\sqrt{3}}.$$

$$\tan 30^\circ = \frac{DB}{DA} = \frac{m}{m\sqrt{3}} = \frac{1}{\sqrt{3}}. \quad \therefore \cot 30^\circ = \sqrt{3}.$$

EXAMPLES.



1. Given $\sin\theta = \frac{3}{5}$; find the other trigonometric functions.

Let BAC be the angle, and BC be perpendicular to AC. Represent BC by 3, AB by 5, and consequently AC by $\sqrt{25 - 9} = 4$.

Then
$$\cos\theta = \frac{AC}{AB} = \frac{4}{5},$$
$$\tan\theta = \frac{BC}{AC} = \frac{3}{4},$$
$$\sec\theta = \frac{AB}{AC} = \frac{5}{4},$$
$$\operatorname{cosec}\theta = \frac{AB}{BC} = \frac{5}{3}.$$

2. Given $\sin\theta=\frac{3}{5}$; find $\tan\theta$ and $\operatorname{cosec}\theta$. *Ans.* $\frac{3}{4}$, $\frac{5}{3}$.

3. Given $\cos\theta=\frac{1}{3}$; find $\sin\theta$ and $\cot\theta$. $\frac{2}{3}\sqrt{2}$, $\frac{1}{2\sqrt{2}}$.

4. Given $\sec\theta=4$; find $\cot\theta$ and $\sin\theta$. $\frac{1}{\sqrt{15}}$, $\frac{\sqrt{15}}{4}$.

5. Given $\tan\theta=\sqrt{3}$; find $\sin\theta$ and $\cos\theta$. $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}$.

6. Given $\sin\theta=\frac{12}{13}$; find $\cos\theta$. $\frac{5}{13}$.

7. Given $\operatorname{cosec}\theta=5$; find $\sec\theta$ and $\tan\theta$. $\frac{5}{2\sqrt{6}}$, $\frac{1}{2\sqrt{6}}$.

8. Given $\sec\theta=\frac{41}{9}$; find $\sin\theta$ and $\cot\theta$. $\frac{40}{41}$, $\frac{9}{40}$.

9. Given $\cot\theta=\frac{2}{\sqrt{5}}$; find $\sin\theta$ and $\sec\theta$. $\frac{\sqrt{5}}{3}$, $\frac{3}{2}$.

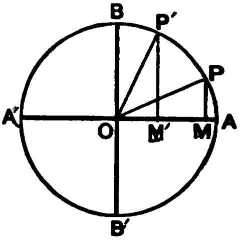
10. Given $\sin\theta=\frac{3}{4}$; find $\cos\theta$, $\tan\theta$, and $\cot\theta$.
$\frac{\sqrt{7}}{4}$, $\frac{3\sqrt{7}}{7}$, $\frac{\sqrt{7}}{3}$.

11. Given $\sin\theta=\frac{b}{c}$; find $\tan\theta$. $\frac{b}{\sqrt{c^2-b^2}}$.

12. Given $\sin\theta=\frac{2mn}{m^2+n^2}$; find $\tan\theta$. $\frac{2mn}{m^2-n^2}$.

28. Reduction of Trigonometric Functions to the 1st Quadrant. — All mathematical tables give the trigonometric functions of angles between 0° and 90° only, but in practice we constantly have to deal with angles greater than 90°. The object of the following six Articles is to show that the trigonometric functions of any angle, positive or negative, can be expressed in terms of the trigonometric functions of an angle less than 90°, so that, if a given angle is greater than 90°, we can find an angle in the 1st quadrant whose trigonometric function has the same absolute value.

29. Functions of Complemental Angles.—Let AA′, BB′ be two diameters of a circle at right angles, and let OP and OP′ be the positions of the radius for any angle $AOP = A$, and its complement $AOP' = 90° - A$ (Art. 12).



Draw PM and P′M′ at right angles to OA.

$$\text{Angle } OP'M' = BOP' = AOP = A.$$

$$\text{Also} \qquad OP = OP'.$$

Hence the triangles OPM and OP′M′ are equal in all respects.

$$\therefore\ P'M' = OM. \qquad \therefore\ \frac{P'M'}{OP'} = \frac{OM}{OP}.$$

$$\therefore\ \sin(90° - A) = \cos AOP = \cos A.$$

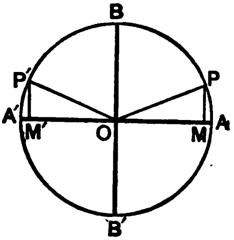
$$\text{Also,} \qquad OM' = PM. \qquad \therefore\ \frac{OM'}{OP'} = \frac{PM}{OP}.$$

$$\therefore\ \cos(90° - A) = \sin AOP = \sin A.$$

$$\text{Similarly,} \quad \tan(90° - A) = \tan AOP' = \frac{P'M'}{OM'} = \frac{OM}{MP} = \cot A.$$

The other relations are obtained by inverting the above.

30. Functions of Supplemental Angles.—Let OP and OP′ be the positions of the radius for any angle $AOP = A$, and its supplement $AOP' = 180° - A$ (Art. 12).



Since $OP = OP'$, and $POA = P'OA'$, the triangles POM and P′OM′ are geometrically equal.

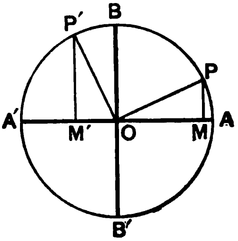
$$\therefore\ \sin(180° - A) = \sin AOP' = \frac{P'M'}{OP'} = \frac{PM}{OP} = \sin A,$$

$$\cos(180° - A) = \cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A,$$

$$\tan(180^\circ - A) = \tan AOP' = \frac{P'M'}{OM'} = \frac{PM}{-OM} = -\tan A.$$

Similarly the other relations may be obtained.

31. To prove $\sin(90^\circ + A) = \cos A$, $\cos(90^\circ + A) = -\sin A$, **and** $\tan(90^\circ + A) = -\cot A$.



Let OP and OP′ be the positions of the radius for any angle AOP = A, and AOP′ = 90° + A.

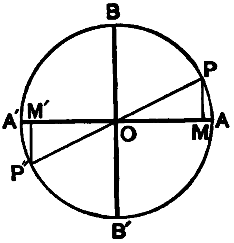
Since OP = OP′, and AOP = P′OB = OP′M′, the triangles POM and P′OM′ are equal in all respects.

$$\therefore \sin(90^\circ + A) = \sin AOP' = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos A,$$

$$\cos(90^\circ + A) = \cos AOP' = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\tan(90^\circ + A) = \tan AOP' = \frac{P'M'}{OM'} = \frac{OM}{-PM} = -\cot A.$$

32. To prove $\sin(180^\circ + A) = -\sin A$, $\cos(180^\circ + A) = -\cos A$, **and** $\tan(180^\circ + A) = \tan A$.



Let the angle AOP = A; then the angle AOP′, measured in the positive direction, = (180° + A).

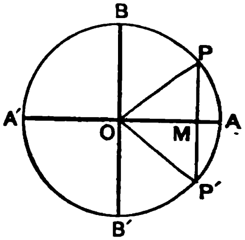
The triangles POM and P′OM′ are equal.

$$\therefore \sin(180^\circ + A) = \sin AOP' = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\cos(180^\circ + A) = \cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A,$$

$$\tan(180^\circ + A) = \tan AOP' = \frac{P'M'}{OM'} = \frac{-PM}{-OM} = \tan A.$$

33. To prove $\sin(-A) = -\sin A$, $\cos(-A) = \cos A$, $\tan(-A) = -\tan A$.



Let OP and OP′ be the positions of the radius for any equal angles AOP and AOP′ measured from the initial line AO in opposite directions. Then if the angle AOP be denoted by A, the numerically equal angle AOP′ will be denoted by $-A$ (Art. 4).

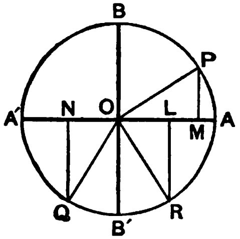
The triangles POM and P′OM are geometrically equal.

$$\therefore \sin(-A) = \sin AOP' = \frac{P'M}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\cos(-A) = \cos AOP' = \frac{OM}{OP'} = \frac{OM}{OP} = \cos A,$$

$$\tan(-A) = \tan AOP' = \frac{P'M}{OM} = \frac{-PM}{OM} = -\tan A.$$

34. To prove $\sin(270^\circ + A) = \sin(270^\circ - A) = -\cos A$, and $\cos(270^\circ + A) = -\cos(270^\circ - A) = \sin A$.



Let the angle $AOP = A$; then the angles AOQ and AOR, measured in the positive direction, $= (270^\circ - A)$ and $(270^\circ + A)$ respectively.

The triangles POM, QON, and ROL are geometrically equal.

$$\therefore \quad RL = QN = OM. \qquad \therefore \frac{RL}{OR} = \frac{QN}{OQ} = \frac{-OM}{OP}.$$

$$\therefore \sin(270^\circ + A) = \sin(270^\circ - A) = -\cos A.$$

Also, $$\frac{OL}{OR} = \frac{-ON}{OQ} = \frac{PM}{OP}.$$

$$\therefore \cos(270^\circ + A) = -\cos(270^\circ - A) = \sin A.$$

35. Table giving the Values of the Functions of Any Angle in Terms of the Functions of an Angle less than 90°. — The foregoing results, and other similar ones, which may be proved in the same manner, are here collected for reference.

Quadrant II.

$$\begin{aligned}
\sin(180^\circ - A) &= \sin A. & \sin(90^\circ + A) &= \cos A.\\
\cos(180^\circ - A) &= -\cos A. & \cos(90^\circ + A) &= -\sin A.\\
\tan(180^\circ - A) &= -\tan A. & \tan(90^\circ + A) &= -\cot A.\\
\cot(180^\circ - A) &= -\cot A. & \cot(90^\circ + A) &= -\tan A.\\
\sec(180^\circ - A) &= -\sec A. & \sec(90^\circ + A) &= -\operatorname{cosec} A.\\
\operatorname{cosec}(180^\circ - A) &= \operatorname{cosec} A. & \operatorname{cosec}(90^\circ + A) &= \sec A.
\end{aligned}$$

Quadrant III.

$$\begin{aligned}
\sin(180^\circ + A) &= -\sin A. & \sin(270^\circ - A) &= -\cos A.\\
\cos(180^\circ + A) &= -\cos A. & \cos(270^\circ - A) &= -\sin A.\\
\tan(180^\circ + A) &= \tan A. & \tan(270^\circ - A) &= \cot A.\\
\cot(180^\circ + A) &= \cot A. & \cot(270^\circ - A) &= \tan A.\\
\sec(180^\circ + A) &= -\sec A. & \sec(270^\circ - A) &= -\operatorname{cosec} A.\\
\operatorname{cosec}(180^\circ + A) &= -\operatorname{cosec} A. & \operatorname{cosec}(270^\circ - A) &= -\sec A.
\end{aligned}$$

Quadrant IV.

$$\begin{aligned}
\sin(360^\circ - A) &= -\sin A. & \sin(270^\circ + A) &= -\cos A.\\
\cos(360^\circ - A) &= \cos A. & \cos(270^\circ + A) &= \sin A.\\
\tan(360^\circ - A) &= -\tan A. & \tan(270^\circ + A) &= -\cot A.\\
\cot(360^\circ - A) &= -\cot A. & \cot(270^\circ + A) &= -\tan A.\\
\sec(360^\circ - A) &= \sec A. & \sec(270^\circ + A) &= \operatorname{cosec} A.\\
\operatorname{cosec}(360^\circ - A) &= -\operatorname{cosec} A. & \operatorname{cosec}(270^\circ + A) &= -\sec A.
\end{aligned}$$

NOTE. — These relations* may be remembered by noting the following rules:

When A is associated with an *even* multiple of 90°, any function of the angle is *numerically* equal to the *same function* of A.

When A is associated with an *odd* multiple of 90°, any function of the angle is numerically equal to the corresponding *cofunction* of the angle A.

The *sign* to be prefixed will depend upon the quadrant to which the angle belongs (Art. 5), regarding A as an acute angle.

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* Although these relations have been proved only in case of A, *an acute angle*, they are true whatever A may be.

[illegible] (270° − A) = − sin A; the angle 270° − A being in the 3d quadrant, and its sine negative in consequence.

[illegible] in the

First quadrant all the functions are *positive*.
Second quadrant all are *negative* except the *sine* and *cosecant*.
Third quadrant all are *negative* except the *tangent* and *cotangent*.
Fourth quadrant all are *negative* except the *cosine* and *secant*.

[illegible] Periodicity of the Trigonometric Functions. — Let AOP be an angle of any magnitude, as in the figure of No. 18; then if OP revolve in the positive or the negative direction through an angle of 360°, it will return to the position from which it started. Hence it is clear from the definitions that the trigonometric functions remain unchanged when the angle is increased or diminished by 360°, or any multiple of 360°. Thus the functions of the angle 400° are the same both in numerical value and in algebraic sign as the functions of the angle of 400° − 360°, *i.e.*, of the angle of 40°. Also the functions of 360° + A are the same in numerical value and in sign as those of A.

In general, if n denote any integer, either positive or negative, *the functions of* $n \times 360° + A$ *are the same as those of* A.

Thus the functions of 1470° = the functions of 30°.

If θ denotes any angle in circular measure, the functions of $(2n\pi + \theta)$ are the same as those of θ. Thus

$$\sin(2n\pi + \theta) = \sin\theta, \ \cos(2n\pi + \theta) = \cos\theta, \text{ etc.}$$

By this proposition we can reduce an angle of any magnitude to an angle less than 360° without changing the values of the functions. It is therefore unnecessary to consider the functions of angles greater than 360°; the formulæ already established *are true for angles of any magnitude whatever.*

EXAMPLES.

1. Express sin 700° in terms of the functions of an acute angle.

$$\sin 700° = \sin(360° + 340°) = \sin 340° = \sin(180° + 160°)$$
$$= \sin 160° = -\sin 20°.$$

Find the values of the following functions:

2. $\sin 810°$, $\sin(-240°)$, $\cos 210°$. *Ans.* $1, \frac{1}{2}\sqrt{3}, -\frac{1}{2}\sqrt{3}$.
3. $\tan(-120°)$, $\cot 420°$, $\cot 510°$. $\sqrt{3}, \frac{1}{\sqrt{3}}, -\sqrt{3}$.
4. $\sin 930°$, $\tan 6420°$. $-\frac{1}{2}, -\sqrt{3}$.
5. $\cot 1035°$, $\operatorname{cosec} 570°$. $-1, -2$.

37. Trigonometric Identities. — A trigonometric identity is an expression which states in the form of an equation a relation which is true for all values of the angle involved. Thus, the relations of Arts. 13 and 23, and all others that may be deduced from them by the aid of the ordinary formulæ of Algebra, are universally true; and are therefore called *identities;* but such relations as $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{1}{3}$, are not *identities.*

EXAMPLES.

1. Prove that $\sec\theta - \tan\theta \cdot \sin\theta = \cos\theta$.

Here
$$\sec\theta - \tan\theta \sin\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\sin\theta. \qquad \text{(Art. 24)}$$
$$= \frac{1 - \sin^2\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta}{\cos\theta} \qquad \text{(Art. 24)}$$
$$= \cos\theta.$$

2. Prove that $\cot\theta - \sec\theta \operatorname{cosec}\theta(1 - 2\sin^2\theta) = \tan\theta$.

$$\cot\theta - \sec\theta \operatorname{cosec}\theta\,(1 - 2\sin^2\theta)$$
$$= \frac{\cos\theta}{\sin\theta} - \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}(1 - 2\sin^2\theta) \qquad \text{(Art. 24)}$$
$$= \frac{\cos^2\theta - 1 + 2\sin^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{\cos^2\theta-(\sin^2\theta+\cos^2\theta)+2\sin^2\theta}{\sin\theta\cos\theta} \qquad \text{(Art. 24)}$$

$$=\frac{\sin^2\theta}{\sin\theta\cos\theta}=\frac{\sin\theta}{\cos\theta}=\tan\theta.$$

NOTE.—It will be observed that in solving these examples we first express the other functions in terms of the sine and cosine, and in most cases the beginner will find this the simplest course. It is generally advisable to begin with the most complicated side and work towards the other.

Prove the following identities:

3. $\cos\theta\tan\theta=\sin\theta.$
4. $\cos\theta=\sin\theta\cot\theta.$
5. $(\tan\theta+\cot\theta)\sin\theta\cos\theta=1.$
6. $(\tan\theta-\cot\theta)\sin\theta\cos\theta=\sin^2\theta-\cos^2\theta.$
7. $\sin^2\theta\div\operatorname{cosec}^2\theta=\sin^4\theta.$
8. $\sec^4\theta-\tan^4\theta=\sec^2\theta+\tan^2\theta.$
9. $(\sin\theta-\cos\theta)^2=1-2\sin\theta\cos\theta.$
10. $1-\tan^4\theta=2\sec^2\theta-\sec^4\theta.$
11. $\dfrac{1+\cos\theta}{1-\cos\theta}=(\operatorname{cosec}\theta+\cot\theta)^2.$
12. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2.$
13. $\sin^4\theta-\cos^4\theta=\sin^2\theta-\cos^2\theta.$
14. $\sin^2\theta+\operatorname{vers}^2\theta=2(1-\cos\theta).$
15. $\cot^2\theta-\cos^2\theta=\cot^2\theta\cos^2\theta.$

EXAMPLES.

In a right triangle ABC (see figure of Art. 15) given:

1. $a=p^2+pq,\ c=q^2+pq$; calculate cot A. *Ans.* $\dfrac{\sqrt{q^2-p^2}}{p}.$
2. $b=lm\div n,\ c=ln\div m$; calculate cosec A. $\dfrac{n^2}{\sqrt{n^4-m^4}}.$

3. $\sin A = \frac{3}{5}$, $c = 20.5$; calculate a. *Ans.* 12.3.

4. Given $\cot \frac{1}{2} A = \tan A$; find A.

5. “ $\sin A = \cos 2A$; find A.

6. “ $\cot A = \tan 6A$; find A.

7. “ $\tan A = \cot 8A$; find A.

8. “ $\sin 2A = \cos 3A$; find A.

9. “ $\sin A = \frac{2}{3}$; find cos A and tan A. $\frac{\sqrt{5}}{3}, \frac{2}{\sqrt{5}}$.

10. “ $\cos A = \frac{4}{5}$; find sin A and tan A. $\frac{3}{5}, \frac{3}{4}$.

11. “ $\operatorname{cosec} A = \frac{4}{3}$; find cos A and tan A. $\frac{\sqrt{7}}{4}, \frac{3}{\sqrt{7}}$.

12. “ $\sin A = \frac{1}{\sqrt{3}}$; find cos A and tan A. $\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{2}}$.

13. “ $\cos A = b$; find tan A and cosec A. $\frac{\sqrt{1-b^2}}{b}, \frac{1}{\sqrt{1-b^2}}$.

14. “ $\sin A = .6$; find cos A and cot A. $\frac{4}{5}, \frac{4}{3}$.

15. “ $\tan A = \frac{4}{5}$; find sin A. $\frac{4}{\sqrt{41}}$.

16. “ $\cot A = \frac{8}{15}$; find sec A and sin A. $\frac{17}{8}, \frac{15}{17}$.

17. “ $\sin A = \frac{12}{13}$; find cos A. $\frac{5}{13}$.

18. “ $\cos A = .28$; find sin A. .96.

19. “ $\tan A = \frac{4}{3}$; find sin A. $\frac{4}{5}$.

20. “ $\sin A = \frac{1}{3}$; find cos A. $\frac{2}{3}\sqrt{2}$.

Express the following functions in terms of the functions of acute angles less than 45°:

21. $\sin 168°$, $\sin 210°$. *Ans.* $\sin 12°$, $-\sin 30°$.
22. $\tan 125°$, $\tan 310°$. $-\cot 35°$, $-\cot 40°$.
23. $\sec 244°$, $\operatorname{cosec} 281°$. $-\operatorname{cosec} 26°$, $-\sec 11°$.
24. $\sec 930°$, $\operatorname{cosec}(-600°)$. $-\sec 30°$, $\sec 30°$.
25. $\cot 460°$, $\sec 299°$. $-\tan 10°$, $\operatorname{cosec} 29°$.
26. $\tan 1400°$, $\cot(-1400°)$. $-\tan 40°$, $\cot 40°$.

Find the values of the following functions:

27. $\sin 120°$, $\sin 135°$, $\sin 240°$. *Ans.* $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{2}}$, $-\frac{\sqrt{3}}{2}$.
28. $\cos 135°$, $\tan 300°$, $\operatorname{cosec} 300°$. $-\frac{1}{\sqrt{2}}$, $-\sqrt{3}$, $-\frac{2}{\sqrt{3}}$.
29. $\sec 315°$, $\cot 330°$, $\tan 780°$. $\sqrt{2}$, $-\sqrt{3}$, $\sqrt{3}$.
30. $\sin 480°$, $\sin 495°$, $\sin 870°$. $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{2}}$, $\frac{1}{2}$.
31. $\tan 1020°$, $\sec 1395°$, $\sin 1485°$. $-\sqrt{3}$, $\sqrt{2}$, $\frac{1}{\sqrt{2}}$.
32. $\tan^3 660°$, $\cos^3 1020°$. $-3\sqrt{3}$, $\frac{1}{8}$.

Prove, drawing a separate figure in each case, that

33. $\sin 340° = \sin(-160°)$.
34. $\sin(-40°) = \sin 220°$.
35. $\cos 320° = -\cos(140°)$.
36. $\cos(-380°) = -\cos 560°$.
37. $\cos 195° = -\cos(-15°)$.
38. $\cos 380° = -\cos 560°$.
39. $\cos(-225°) = -\cos(-45°)$.
40. $\cos 1005° = -\cos 1185°$.

Prove the following identities:

41. $(\sin^2\theta + \cos^2\theta)^2 = 1.$

42. $(\sin^2\theta - \cos^2\theta)^2 = 1 - 4\cos^2\theta + 4\cos^4\theta.$

43. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta.$

44. $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1.$

45. $(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta) = 1.$

46. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta).$

47. $\sin^6\theta + \cos^6\theta = \sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta.$

48. $\sin^2\theta\tan^2\theta + \cos^2\theta\cot^2\theta = \tan^2\theta + \cot^2\theta - 1.$

49. $\sin\theta\tan^2\theta + \operatorname{cosec}\theta\sec^2\theta - 2\tan\theta\sec\theta = \operatorname{cosec}\theta - \sin\theta.$

50. $\cos^3\theta - \sin^3\theta = (\cos\theta - \sin\theta)(1 + \sin\theta\cos\theta).$

51. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta.$

52. $\tan\alpha + \tan\beta = \tan\alpha\tan\beta(\cot\alpha + \cot\beta).$

53. $\cot\alpha + \tan\beta = \cot\alpha\tan\beta(\tan\alpha + \cot\beta).$

54. $1 - \sin\alpha = (1 + \sin\alpha)(\sec\alpha - \tan\alpha)^2.$

55. $1 + \cos\alpha = (1 - \cos\alpha)(\operatorname{cosec}\alpha + \cot\alpha)^2.$

56. $(1 + \sin\alpha + \cos\alpha)^2 = 2(1 + \sin\alpha)(1 + \cos\alpha).$

57. $(1 - \sin\alpha - \cos\alpha)^2(1 + \sin\alpha + \cos\alpha)^2 = 4\sin^2\alpha\cos^2\alpha.$

58. $2\operatorname{vers}\alpha - \operatorname{vers}^2\alpha = \sin^2\alpha.$

59. $\operatorname{vers}\alpha(1 + \cos\alpha) = \sin^2\alpha.$

CHAPTER III.

TRIGONOMETRIC FUNCTIONS OF TWO ANGLES.

38. Fundamental Formulæ. — We now proceed to express the trigonometric functions of the sum and difference of two angles in terms of the trigonometric functions of the angles themselves.

The *fundamental formulæ* first to be established are the following:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad . \quad . \quad . \quad . \qquad (1)$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad . \quad . \quad . \quad . \qquad (2)$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \quad . \quad . \quad . \quad . \qquad (3)$$

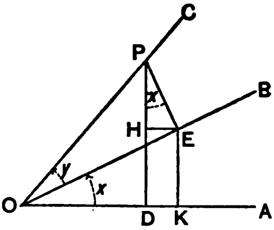
$$\cos(x-y) = \cos x \cos y + \sin x \sin y \quad . \quad . \quad . \quad . \qquad (4)$$

Note. — Here x and y are angles; so that $(x+y)$ and $(x-y)$ are also angles. Hence, sin $(x+y)$ is the sine of an angle, and is not the same as sin x + sin y.
Sin $(x+y)$ is a single fraction.
Sin x + sin y is the sum of two fractions.

39. To prove that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

and $$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$



Let the angle AOB $= x$, and the angle BOC $= y$; then the angle AOC $= x + y$.

In OC, *the bounding line of the angle* $(x+y)$, take any point P, and draw PD, PE, perpendicular to OA and OB, respectively; draw EH, EK, perpendicular to PD and OA.

Then angle EPH = 90° − HEP = HEO = AOE = x.

$$\sin(x+y)=\frac{\text{DP}}{\text{OP}}=\frac{\text{EK}+\text{PH}}{\text{OP}}=\frac{\text{EK}}{\text{OP}}+\frac{\text{PH}}{\text{OP}}$$

$$=\frac{\text{EK}}{\text{OE}}\cdot\frac{\text{OE}^{*}}{\text{OP}}+\frac{\text{PH}}{\text{PE}}\cdot\frac{\text{PE}}{\text{OP}}$$

$$=\sin x\cos y+\cos x\sin y.$$

$$\cos(x+y)=\frac{\text{OD}}{\text{OP}}=\frac{\text{OK}-\text{HE}}{\text{OP}}=\frac{\text{OK}}{\text{OP}}-\frac{\text{HE}}{\text{OP}}$$

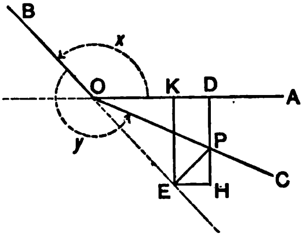
$$=\frac{\text{OK}}{\text{OE}}\cdot\frac{\text{OE}}{\text{OP}}-\frac{\text{HE}}{\text{PE}}\cdot\frac{\text{PE}}{\text{OP}}$$

$$=\cos x\cos y-\sin x\sin y.$$

Note. — These two formulæ have been obtained by a construction in which $x+y$ is an acute angle; but the proof is perfectly general, and applies to angles of any magnitude whatever, by paying due regard to the algebraic signs. For example,

Let AOB = x, as before, and BOC = y; then AOC, measured in the positive direction, is the angle $x+y$.

In OC, take any point P, and draw PD, PE, perpendicular to OA and OB produced; draw EH and EK perpendicular to PD and OA.



Then, angle EOK = 180° − x;

EPH = EOK = 180° − x;

and COE = y − 180°.

$$\sin(x+y)=-\frac{\text{DP}}{\text{OP}}=-\frac{\text{EK}-\text{PH}}{\text{OP}}=-\frac{\text{EK}}{\text{OP}}+\frac{\text{PH}}{\text{OP}}$$

$$=-\frac{\text{EK}}{\text{OE}}\cdot\frac{\text{OE}}{\text{OP}}+\frac{\text{PH}}{\text{PE}}\cdot\frac{\text{PE}}{\text{OP}}$$

$$=-\sin(180^\circ-x)\cos(y-180^\circ)+\cos(180^\circ-x)\sin(y-180^\circ)$$

$$=\sin x\cos y+\cos x\sin y. \qquad \text{(Art. 35)}$$

$$\cos(x+y)=\frac{\text{OD}}{\text{OP}}=\frac{\text{OK}+\text{EH}}{\text{OP}}=\frac{\text{OK}}{\text{OP}}+\frac{\text{EH}}{\text{OP}}$$

$$=\frac{\text{OK}}{\text{OE}}\cdot\frac{\text{OE}}{\text{OP}}+\frac{\text{EH}}{\text{PE}}\cdot\frac{\text{PE}}{\text{OP}}$$

---

[*] The introduced line OE is the *only* line in the figure which is at once a side of two right triangles (OEK and OEP) into which EK and OP enter. A similar remark applies to PE.

$$= \cos(180° - x)\cos(y - 180°) + \sin(180° - x)\sin(y - 180°)$$

$$= \cos x \cos y - \sin x \sin y. \qquad \text{(Art. 35)}$$

The student should notice that the words of the two proofs are very nearly the same.

40. To prove that

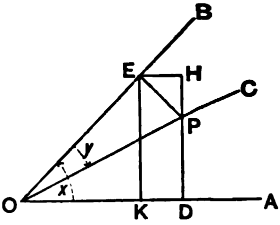
$$\sin(x - y) = \sin x \cos y - \cos x \sin y,$$

and

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

Let the angle AOB be denoted by x, and COB by y; then the angle AOC $= x - y$.

In OC take any point P*, and draw PD, PE, perpendicular to OA and OB respectively; draw EH, EK, perpendicular to PD and OA respectively.



Then the angle EPH $= 90° -$ HEP $=$ BEH $=$ AOB $= x$.

$$\sin(x - y) = \frac{DP}{OP} = \frac{EK - HP}{OP} = \frac{EK}{OP} - \frac{HP}{OP}$$

$$= \frac{EK}{OE} \cdot \frac{OE}{OP} - \frac{HP}{PE} \cdot \frac{PE}{OP}$$

$$= \sin x \cos y - \cos x \sin y.$$

$$\cos(x - y) = \frac{OD}{OP} = \frac{OK + EH}{OP} = \frac{OK}{OP} + \frac{EH}{OP}$$

$$= \frac{OK}{OE} \cdot \frac{OE}{OP} + \frac{EH}{PE} \cdot \frac{PE}{OP}$$

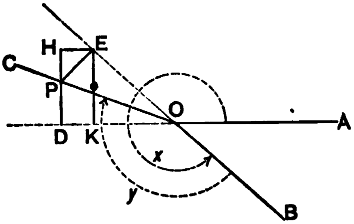
$$= \cos x \cos y + \sin x \sin y.$$

NOTE 1. — The sign in the expression of the sine is the *same* as it is in the angle expanded; in the cosine it is the *opposite*.

---

* **P is taken in the line bounding the angle under consideration; *i.e.*, AOC.**

Note 2.—In this proof the angle $x-y$ is acute; but the proof, like the one given in Art. 39, applies to angles of any magnitude whatever. For example,



Let AOB, measured in the positive direction, $=x$, and $\text{BOC}=y$. Then $\text{AOC}=x-y$.

In OC take any point P, and draw PD, PE, perpendicular to OA and OB produced: draw EH, EK, perpendicular to DP and AO produced.

Then,

$$\text{angle EPH}=\text{EOK}=\text{AOB}=360^\circ-x,$$

$$\text{and}\quad \text{POE}=180^\circ-y.$$

$$\sin(x-y)=\frac{\text{PD}}{\text{OP}}=\frac{\text{EK}-\text{HP}}{\text{OP}}$$

$$=\frac{\text{EK}}{\text{OE}}\cdot\frac{\text{OE}}{\text{OP}}-\frac{\text{HP}}{\text{PE}}\cdot\frac{\text{PE}}{\text{OP}}$$

$$=\sin(360^\circ-x)\cos(180^\circ-y)-\cos(360^\circ-x)\sin(180^\circ-y)$$

$$=(-\sin x)(-\cos y)-\cos x\sin y$$

$$=\sin x\cos y-\cos x\sin y.$$

$$\cos(x-y)=-\frac{\text{OD}}{\text{OP}}=-\frac{\text{OK}+\text{HE}}{\text{OP}}$$

$$=-\frac{\text{OK}}{\text{OE}}\cdot\frac{\text{OE}}{\text{OP}}-\frac{\text{HE}}{\text{PE}}\cdot\frac{\text{PE}}{\text{OP}}$$

$$=-\cos(360^\circ-x)\cos(180^\circ-y)-\sin(360^\circ-x)\sin(180^\circ-y)$$

$$=(-\cos x)(-\cos y)-(-\sin x)\sin y$$

$$=\cos x\cos y+\sin x\sin y.$$

Note 3.—The four fundamental formulæ just proved are very important, *and must be committed to memory*. It will be convenient to refer to them as the 'x, y' formulæ. From any one of them, all the others can be deduced in the following manner:

Thus, from $\cos(x-y)$ to deduce $\sin(x+y)$. We have

$$\cos(x-y)=\cos x\cos y+\sin x\sin y\quad\ldots\quad(1)$$

Substitute $90^\circ-x$ for x in (1), and it becomes

$$\cos\{90^\circ-(x+y)\}=\cos(90^\circ-x)\cos y+\sin(90^\circ-x)\sin y.$$

$$\therefore\ \sin(x+y)=\sin x\cos y+\cos x\sin y.\qquad(\text{Art. }29)$$

The student should make the substitutions indicated below, and satisfy himself that the corresponding results follow:

From

$\sin(x+y)$	to	deduce	$\cos(x+y)$	substitute	$(90°+x)$ for x.
"	"	"	$\cos(x-y)$	"	$(90°-x)$ for x.
$\cos(x+y)$	"	"	$\sin(x+y)$	"	$(90°+x)$ for x.
"	"	"	$\sin(x-y)$	"	$(90°-x)$ for x.
"	"	"	$\cos(x-y)$	"	$-y$ for y.
etc.			etc.		etc.

EXAMPLES.

1. To find the value of $\sin 15°$.

$$\begin{aligned}\sin 15° &= \sin(45°-30°)\\ &= \sin 45° \cos 30° - \cos 45° \sin 30°\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}.\end{aligned}$$

2. Show that $\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

3. Show that $\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

4. Show that $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

5. If $\sin x = \dfrac{3}{5}$, and $\cos y = \dfrac{5}{13}$, find $\sin(x+y)$ and $\cos(x-y)$.

Ans. $\dfrac{63}{65}$, and $\dfrac{56}{65}$.

6. If $\sin x = \dfrac{1}{2}$, and $\cos y = \dfrac{1}{2}$, find $\sin(x+y)$ and $\cos(x-y)$.

Ans. 1, and $\dfrac{\sqrt{3}}{2}$.

41. Formulæ for the Transformation of Sums into Products. — From the four fundamental formulæ of Arts. 39 and 40 we have, by addition and subtraction, the following:

$$\sin(x+y)+\sin(x-y)=2\sin x\cos y \quad . \quad . \quad . \quad (1)$$

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y \quad . \quad . \quad . \quad (2)$$

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y \quad . \quad . \quad . \quad (3)$$

$$\cos(x-y)-\cos(x+y)=2\sin x\sin y \quad . \quad . \quad . \quad (4)$$

These formulæ are useful in proving identities by transforming products into terms of first degree. They enable us, when read from right to left, to replace the *product* of a *sine or a cosine into a sine or a cosine* by half the *sum* or half the *difference* of two such ratios.

Let $x+y=\text{A}$, and $x-y=\text{B}$.

$$\therefore\ x=\tfrac{1}{2}(\text{A}+\text{B}), \text{ and } y=\tfrac{1}{2}(\text{A}-\text{B}).$$

Substituting these values in the above formulæ, and putting, for the sake of uniformity of notation, x, y instead of A, B, we get

$$\sin x+\sin y=2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \quad . \quad . \quad (5)$$

$$\sin x-\sin y=2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \quad . \quad . \quad (6)$$

$$\cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \quad . \quad . \quad (7)$$

$$\cos y-\cos x=2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \quad . \quad . \quad (8)$$

The formulæ are of great importance in mathematical investigations (especially in computations by logarithms); they enable us to express *the sum or the difference* of two sines or two cosines in the form of a *product*. The student is recommended to become familiar with them, and to commit the following enunciations to memory:

Of any two angles, the

Sum of the sines $= 2\sin\frac{1}{2}\text{sum}\cdot\cos\frac{1}{2}\text{diff.}$

Diff. " " " $= 2\cos\frac{1}{2}\text{sum}\cdot\sin\frac{1}{2}\text{diff.}$

Sum of the cosines $= 2\cos\frac{1}{2}\text{sum}\cdot\cos\frac{1}{2}\text{diff.}$

Diff. " " " $= 2\sin\frac{1}{2}\text{sum}\cdot\sin\frac{1}{2}\text{diff.}$

EXAMPLES.

1. $\sin 5x\cos 3x = \frac{1}{2}(\sin 8x + \sin 2x).$

For, $\sin 5x\cos 3x = \frac{1}{2}\{\sin(5x+3x) + \sin(5x-3x)\}$
$= \frac{1}{2}(\sin 8x + \sin 2x).$

2. Prove $\sin\theta\sin 3\theta = \frac{1}{2}(\cos 2\theta - \cos 4\theta).$
3. " $2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi).$
4. " $2\sin 2\theta\cos 3\phi = \sin(2\theta+3\phi) + \sin(2\theta-3\phi).$
5. " $\sin 60° + \sin 30° = 2\sin 45°\cos 15°.$
6. " $\sin 40° - \sin 10° = 2\cos 25°\sin 15°.$
7. " $\sin 10\theta + \sin 6\theta = 2\sin 8\theta\cos 2\theta.$
8. " $\sin 8\alpha - \sin 4\alpha = 2\cos 6\alpha\sin 2\alpha.$
9. " $\sin 3x + \sin x = 2\sin 2x\cos x.$
10. " $\sin 3x - \sin x = 2\cos 2x\sin x.$
11. " $\sin 4x + \sin 2x = 2\sin 3x\cos x.$

42. Useful Formulæ. — The following formulæ, which are of frequent use, may be deduced by taking the quotient of each pair of the formulæ (5) to (8) of Art. 41 as follows:

$$1.\quad \frac{\sin x + \sin y}{\sin x - \sin y} = \frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)}$$
$$= \tan\frac{1}{2}(x+y)\cot\frac{1}{2}(x-y)$$
$$= \frac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}. \qquad \text{(Art. 24)}$$

The following may be proved by the student in a similar manner:

$$2.\quad \frac{\sin x + \sin y}{\cos x + \cos y} = \tan\frac{1}{2}(x+y),$$

3. $\dfrac{\sin x+\sin y}{\cos y-\cos x}=\cot\frac{1}{2}(x-y)$,

4. $\dfrac{\sin x-\sin y}{\cos x+\cos y}=\tan\frac{1}{2}(x-y)$,

5. $\dfrac{\sin x-\sin y}{\cos y-\cos x}=\cot\frac{1}{2}(x+y)$,

6. $\dfrac{\cos x+\cos y}{\cos y-\cos x}=\cot\frac{1}{2}(x+y)\cot\frac{1}{2}(x-y)$.

43. The Tangent of the Sum and Difference of Two Angles. —Expressions for the value of $\tan(x+y)$, $\tan(x-y)$, etc., may be established *geometrically.* It is simpler, however, to deduce them from the formulæ already established, as follows:

Dividing the first of the 'x, y' formulæ by the second, we have, by Art. 23,

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}.$$

Dividing both terms of the fraction by $\cos x\cos y$,

$$\tan(x+y)=\frac{\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}}$$

$$=\frac{\tan x+\tan y}{1-\tan x\tan y}\text{ (Art. 23)}\quad\ldots\quad(1)$$

In the same manner may be derived

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}\quad\ldots\ldots\ldots\quad(2)$$

Also, $$\cot(x+y)=\frac{\cot x\cot y-1}{\cot x+\cot y}\quad\ldots\ldots\ldots\quad(3)$$

and $$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}\quad\ldots\ldots\ldots\quad(4)$$

EXERCISES.

Prove the following:

1. $$\tan(x+45^\circ)=\frac{\tan x+1}{1-\tan x}.$$

2. $$\tan(x-45^\circ)=\frac{\tan x-1}{1+\tan x}.$$

3. $$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\tan x+\tan y}{\tan x-\tan y}.$$

4. $$\frac{\cos(x-y)}{\cos(x+y)}=\frac{\tan x\tan y+1}{1-\tan x\tan y}.$$

5. $$\sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y$$
$$=\cos^2 y-\cos^2 x.$$

6. $$\cos(x+y)\cos(x-y)=\cos^2 x-\sin^2 y$$
$$=\cos^2 y-\sin^2 x.$$

7. $$\tan x\pm\tan y=\frac{\sin(x\pm y)}{\cos x\cos y}.$$

8. $$\cot x\pm\cot y=\frac{\sin(y\pm x)}{\sin x\sin y}.$$

9. $$\frac{\sin 2x}{\sin x}-\frac{\cos 2x}{\cos x}=\sec x.$$

10. If $\tan x=\frac{1}{2}$ and $\tan y=\frac{1}{4}$, prove that $\tan(x+y)=\frac{6}{7}$, and $\tan(x-y)=\frac{2}{9}$.

11. Prove that $\tan 15^\circ=2-\sqrt{3}$.

12. If $\tan x=\frac{5}{6}$, and $\tan y=\frac{1}{11}$, prove that $\tan(x+y)=1$. What is $(x+y)$ in this case?

44. Formulæ for the Sum of Three or More Angles. — Let x, y, z be any three angles; we have by Art. 39,

$$\sin(x+y+z) = \sin(x+y)\cos z + \cos(x+y)\sin z$$

$$= \sin x\cos y\cos z + \cos x\sin y\cos z$$

$$+ \cos x\cos y\sin z - \sin x\sin y\sin z \quad . \quad . \quad (1)$$

In like manner,

$$\cos(x+y+z) = \cos x\cos y\cos z - \sin x\sin y\cos z$$

$$- \sin x\cos y\sin z - \cos x\sin y\sin z \quad . \quad . \quad (2)$$

Dividing (1) by (2), and reducing by dividing both terms of the fraction by $\cos x\cos y\cos z$, we get

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1 - \tan x\tan y - \tan y\tan z - \tan z\tan x} \quad (3)$$

EXAMPLES.

1. Prove that $\sin x + \sin y + \sin z - \sin(x+y+z)$

$$= 4\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(y+z)\sin\tfrac{1}{2}(z+x).$$

By (6) of Art. 41 we have

$$\sin x - \sin(x+y+z) = -2\cos\tfrac{1}{2}(2x+y+z)\sin\tfrac{1}{2}(y+z),$$

and $\qquad \sin y + \sin z = 2\sin\tfrac{1}{2}(y+z)\cos\tfrac{1}{2}(y-z).$

$\therefore\ \sin x + \sin y + \sin z - \sin(x+y+z)$

$$=2\sin\tfrac{1}{2}(y+z)\cos\tfrac{1}{2}(y-z) - 2\cos\tfrac{1}{2}(2x+y+z)\sin\tfrac{1}{2}(y+z)$$

$$=2\sin\tfrac{1}{2}(y+z)\left\{\cos\tfrac{1}{2}(y-z) - \cos\tfrac{1}{2}(2x+y+z)\right\}$$

$$=2\sin\tfrac{1}{2}(y+z)\,2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x+z)$$

$$=4\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(y+z)\sin\tfrac{1}{2}(z+x).$$

Prove the following:

2. $\cos x + \cos y + \cos z + \cos(x+y+z)$

$$= 4\cos\tfrac{1}{2}(y+z)\cos\tfrac{1}{2}(z+x)\cos\tfrac{1}{2}(x+y).$$

3. $\sin(x+y-z) = \sin x \cos y \cos z + \cos x \sin y \cos z$
$- \cos x \cos y \sin z + \sin x \sin y \sin z.$

4. $\sin x + \sin y - \sin z - \sin(x+y-z)$
$= 4 \sin \tfrac{1}{2}(x-z) \sin \tfrac{1}{2}(y-z) \sin \tfrac{1}{2}(x+y).$

5. $\sin(y-z) + \sin(z-x) + \sin(x-y)$
$+ 4 \sin \tfrac{1}{2}(y-z) \sin \tfrac{1}{2}(z-x) \sin \tfrac{1}{2}(x-y) = 0.$

45. Functions of Double Angles. — To express the trigonometric functions of the angle $2x$ in terms of those of the angle x.

Put $y = x$ in (1) of Art. 38, and it becomes

$$\sin 2x = \sin x \cos x + \cos x \sin x,$$

or

$$\sin 2x = 2 \sin x \cos x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put $y = x$ in (2) of Art. 38, and it becomes

$$\cos 2x = \cos^2 x - \sin^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$= 1 - 2\sin^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$= 2\cos^2 x - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Put $y = x$ in (1) and (3) of Art. 43, and they become

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cot 2x = \frac{\cot^2 x - 1}{2\cot x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Transposing 1 in (4), and dividing it into (1), we have

$$\frac{\sin 2x}{1 + \cos 2x} = \tan x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

NOTE. — These seven formulæ are very important. The student must notice that x is *any* angle, and therefore these formulæ will be true whatever we put for x. Thus, if we write $\frac{x}{2}$ for x, we get

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \quad \text{. . . . .} \quad (8)$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \quad \text{. . . . .} \quad (9)$$

or

$$= 1 - 2\sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 \quad \text{. . . . .} \quad (10)$$

and so on.

EXAMPLES.

Prove the following:

1. $2\operatorname{cosec} 2x = \sec x \operatorname{cosec} x.$

2. $\dfrac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x - 2} = \sec 2x.$

3. $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x.$

4. $\dfrac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x.$

5. $\tan x + \cot x = 2\operatorname{cosec} 2x.$

6. $\cot x - \tan x = 2\cot 2x.$

7. $\dfrac{\sin x}{1 + \cos x} = \tan\dfrac{x}{2}.$

8. $\dfrac{\sin x}{1 - \cos x} = \cot\dfrac{x}{2}.$

9. Given $\sin 45° = \dfrac{1}{\sqrt{2}}$; find $\tan 22\frac{1}{2}°$. *Ans.* $\sqrt{2} - 1.$

10. Given $\tan x = \dfrac{3}{4}$; find $\tan 2x$, and $\sin 2x$. $\dfrac{24}{7}, \dfrac{24}{25}.$

46. To Express the Functions of $3x$ in Terms of the Functions of x.

Put $y = 2x$ in (1) of Art. 38, and it becomes

$$\begin{aligned}\sin 3x &= \sin(2x+x)\\ &= \sin 2x\cos x + \cos 2x \sin x\\ &= 2\sin x\cos^2 x + (1-2\sin^2 x)\sin x \qquad \text{(Art. 45)}\\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x.\end{aligned}$$

$$\begin{aligned}\cos 3x &= \cos(2x+x)\\ &= \cos 2x\cos x - \sin 2x\sin x\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

$$\begin{aligned}\tan 3x &= \tan(2x+x)\\ &= \frac{\tan 2x + \tan x}{1-\tan 2x\tan x}\\ &= \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}.\end{aligned}$$

EXAMPLES.

Prove the following:

1. $\dfrac{\sin 3x}{\sin x} = 2\cos 2x + 1.$

2. $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x.$

3. $\dfrac{\sin 3x + \cos 3x}{\cos x - \sin x} = 2\sin 2x + 1.$

4. $\dfrac{1}{\tan 3x - \tan x} + \dfrac{1}{\cot x - \cot 3x} = \cot 2x.$

47. Functions of Half an Angle. — To express the functions of $\frac{x}{2}$ in terms of the functions of x.

Since $$\cos x = 1 - 2\sin^2\frac{x}{2},$$

or $$= 2\cos^2\frac{x}{2} - 1 \quad . \; . \; . \quad \text{[Art. 45, (10)]}$$

$$\therefore \sin^2\frac{x}{2} = \frac{1-\cos x}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and $$\cos^2\frac{x}{2} = \frac{1+\cos x}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Or $$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

and $$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

By division, $$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \pm\frac{1-\cos x}{\sin x} \quad . \; . \quad (5)$$

By formulæ (3), (4), and (5) the functions of half an angle may be found when the cosine of the whole angle is given.

48. Find the Values of the Functions of 22½°.—In (3), (4), and (5) of Art. 47, put $x = 45°$. Then

$$\sin 22\tfrac{1}{2}° = \sqrt{\frac{1-\cos 45°}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2},$$

$$\cos 22\tfrac{1}{2}° = \sqrt{\frac{1+\cos 45°}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\tan 22\tfrac{1}{2}° = \frac{1-\cos 45°}{\sin 45°} = \sqrt{2} - 1.$$

Since 22½° is an acute angle, its functions are all positive. The above results are also the cosine, sine, and cotangent respectively of 67½°, since the latter is the complement of 22½° (Art. 15).

49. Find the Sine and Cosine of 18°.

Let $x = 18°$; then $2x = 36°$, and $3x = 54°$.

$$\therefore 2x + 3x = 90°.$$

$$\therefore \sin 2x = \cos 3x \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(Art. 15)}$$

$$\therefore 2 \sin x \cos x = 4 \cos^3 x - 3 \cos x \quad . \quad . \qquad \text{(Art. 46)}$$

or

$$2 \sin x = 4 \cos^2 x - 3$$

$$= 1 - 4 \sin^2 x.$$

Solving the quadratic, and taking the upper sign, since $\sin 18°$ must be positive, we get

$$\sin 18° = \frac{\sqrt{5} - 1}{4}.$$

Also, $\cos 18° = \sqrt{1 - \sin^2 18°} = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4}.$

Hence we have also the sine and cosine of 72° (Art. 15).

50. Inverse Trigonometric Functions. — The equation $\sin \theta = x$ means that θ is the angle whose sine is x; this may be written $\theta = \sin^{-1} x$, where $\sin^{-1} x$ is an abbreviation for *the angle* (*or arc*) *whose sine is* x.

So the symbols $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} y$, are read "the angle (or arc) whose cosine is x," "the angle (or arc) whose tangent is x," and "the angle (or arc) whose secant is y." These angles are spoken of as being *the inverse sine of* x, *the inverse cosine of* x, *the inverse tangent of* x, and *the inverse secant of* y, respectively. Such expressions are called *inverse trigonometric functions.*

NOTE. — The student must be careful to notice that -1 is not an exponent, $\sin^{-1} x$ is not $(\sin x)^{-1}$, which $= \dfrac{1}{\sin x}$.

Notice also that $\sin^{-1} \dfrac{\sqrt{3}}{2} = \cos^{-1} \dfrac{1}{2}$ is not an *identity*, but is true only for the particular angle 60°.

This notation is only *analogous* to the use of exponents in multiplication, where we have $a^{-1}a = a^0 = 1$. Thus, $\cos^{-1}(\cos x) = x$, and $\sin(\sin^{-1}x) = x$; that is, $\cos^{-1}$ is inverse to cos, and applied to it annuls it; and so for other functions.

The French method of writing inverse functions is *arc sin x*, *arc cos x*, *arc tan x*, and so on.

EXAMPLES.

1. Show that 30° is one value of $\sin^{-1}\frac{1}{2}$.

We know that $\sin 30° = \frac{1}{2}$. $\therefore$ 30° is an angle whose sine is $\frac{1}{2}$; or $30° = \sin^{-1}\frac{1}{2}$.

2. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$.

$\tan^{-1}\frac{1}{2}$ is one of the angles whose tangent is $\frac{1}{2}$, and $\tan^{-1}\frac{1}{3}$ is one of the angles whose tangent is $\frac{1}{3}$.

Let $\alpha = \tan^{-1}\frac{1}{2}$, and $\beta = \tan^{-1}\frac{1}{3}$;

then $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$.

Now $$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \quad . \quad . \quad . \quad \text{(Art. 43)}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1.$$

But $\tan 45° = 1$, $\therefore \alpha + \beta = 45°$;

that is, $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$.

Therefore 45° is one value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

3. Prove that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x + y}{1 - xy}$.

Let $\tan^{-1}x = \text{A}$. $\therefore \tan\text{A} = x$.

$\tan^{-1}y = \text{B}$. $\therefore \tan\text{B} = y$.

Now $$\tan(\text{A} + \text{B}) = \frac{\tan\text{A} + \tan\text{B}}{1 - \tan\text{A}\tan\text{B}}$$

$$= \frac{x + y}{1 - xy}.$$

$$\therefore\ A+B=\tan^{-1}\frac{x+y}{1-xy}.$$

$$\therefore\ \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}.$$

Any relations which have been established among the trigonometric functions may be expressed by means of the inverse notation. Thus, we know that

4. $$\cos x=\sqrt{1-\sin^2 x}.$$

This may be written $x=\cos^{-1}\sqrt{1-\sin^2 x}$ . . (1)

Put $\sin x=\theta$; then $x=\sin^{-1}\theta$.

Thus (1) becomes $\sin^{-1}\theta=\cos^{-1}\sqrt{1-\theta^2}$.

5. By Art. 45, $\cos 2\theta=2\cos^2\theta-1$,

which may be written $2\theta=\cos^{-1}(2\cos^2\theta-1)$.

Put $\cos\theta=x$. $\therefore\ 2\cos^{-1}x=\cos^{-1}(2x^2-1)$.

6. By Art. 45, $\sin 2\theta=2\sin\theta\cos\theta$,

which may be written $2\theta=\sin^{-1}(2\sin\theta\cos\theta)$.

Put $\sin\theta=x$. $\therefore\ 2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$.

7. Prove $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$.

8. " $\tan^{-1}x=\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}=\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$.

9. " $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$.

10. " $\sin(2\sin^{-1}x)=2x\sqrt{1-x^2}$.

11. " $\tan^{-1}\dfrac{5}{7}+\tan^{-1}\dfrac{1}{6}=\dfrac{\pi}{4}$.

51. Solution of Trigonometric Equations. — A **Trigonometric Equation** is an equation in which the unknown quantities involve trigonometric functions.

The *solution* of a trigonometric equation is the process of finding the values of the unknown quantity which satisfy the equation. As in Algebra, we may have two or more simultaneous equations, the number of angles involved being equal to the number of equations.

EXAMPLES.

1. Solve $\sin\theta = \frac{1}{2}$.

This is a trigonometric equation. To solve it we must find some angle whose sine is $\frac{1}{2}$. We know that $\sin 30° = \frac{1}{2}$.

Therefore, if 30° be put for θ, the equation is satisfied.

$\therefore\ \theta = 30°$ is a solution of the equation.

2. Solve $\cos\theta + \sec\theta = \frac{5}{2}$.

The usual method of solution is to express all the functions in terms of one of them.

Thus, we put $\frac{1}{\cos\theta}$ for $\sec\theta$, and get

$$\cos\theta + \frac{1}{\cos\theta} = \frac{5}{2}.$$

This is an equation in which θ, and therefore $\cos\theta$, is unknown. We proceed to solve the equation algebraically just as we should if x occupied the place of $\cos\theta$, thus:

$$\cos^2\theta - \frac{5}{2}\cos\theta = -1.$$

$$\therefore\ \cos\theta = \frac{5}{4} \pm \frac{3}{4}$$

$$= 2 \text{ or } \frac{1}{2}.$$

The value 2 is inadmissible, for there is no angle whose cosine is numerically greater than 1 (Art. 21).

$$\therefore\ \cos\theta = \frac{1}{2}.$$

But $$\cos 60^\circ = \frac{1}{2}.$$

$$\therefore\ \cos\theta = \cos 60^\circ.$$

Therefore *one* value of θ which satisfies the equation is 60°.

3. Solve $\operatorname{cosec}\theta - \cot^2\theta + 1 = 0$.

We have $$\operatorname{cosec}\theta - (\operatorname{cosec}^2\theta - 1) + 1 = 0. \quad . \quad \text{(Art. 23)}$$

$$\operatorname{cosec}^2\theta - \operatorname{cosec}\theta = 2.$$

$$\therefore\ \operatorname{cosec}\theta = \frac{1}{2} \pm \frac{3}{2}$$

$$= 2 \text{ or } -1.$$

But $$\operatorname{cosec} 30^\circ = 2.$$

$$\therefore\ \operatorname{cosec}\theta = \operatorname{cosec} 30^\circ.$$

Therefore 30° is one value of θ which satisfies the equation.

Find a value of θ which will satisfy the following equations:

4. $\cos\theta = \cos 2\theta$. *Ans.* $\frac{2}{3}\pi$.

5. $2\cos\theta = \sec\theta$. 45°.

6. $4\sin\theta - 3\operatorname{cosec}\theta = 0$. 60°.

7. $4\cos\theta = 3\sec\theta$. 30°.

8. $3\sin\theta - 2\cos^2\theta = 0$. 30°.

9. $\sqrt{2}\sin\theta = \tan\theta$. 0° or 45°.

10. $\tan\theta = 3\cot\theta$. 60°.

11. $\tan\theta + 3\cot\theta = 4$. 45°.

12. $\cos\theta + \cos 3\theta + \cos 5\theta = 0$. $\frac{\pi}{2}, \frac{2\pi}{3}$.

52. Table of Useful Formulæ. — The following is a list of important formulæ proved in this chapter, and summed up for the convenience of the student:

1. $\sin(x+y) = \sin x \cos y + \cos x \sin y$ . . (Art. 39)
2. $\cos(x+y) = \cos x \cos y - \sin x \sin y$.
3. $\sin(x-y) = \sin x \cos y - \cos x \sin y$ . . (Art. 40)
4. $\cos(x-y) = \cos x \cos y + \sin x \sin y$.
5. $2\sin x \cos y = \sin(x+y) + \sin(x-y)$. (Art. 41)
6. $2\cos x \sin y = \sin(x+y) - \sin(x-y)$.
7. $2\cos x \cos y = \cos(x+y) + \cos(x-y)$.
8. $2\sin x \sin y = \cos(x-y) - \cos(x+y)$.
9. $\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.
10. $\sin x - \sin y = 2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.
11. $\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.
12. $\cos y - \cos x = 2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.
13. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}$ . . . . . (Art. 42)
14. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$. . . . . (Art. 43)
15. $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$.
16. $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$.
17. $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$.
18. $\tan(x \pm 45^\circ) = \dfrac{\tan x \pm 1}{1 \mp \tan x}$ . . . . (Art. 43)

19. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x.$

20. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y = \cos^2 y - \sin^2 x.$

21. $\tan x \pm \tan y = \dfrac{\sin(x \pm y)}{\cos x \cos y}.$

22. $\cot x \pm \cot y = \dfrac{\sin(y \pm x)}{\sin x \sin y}.$

23. $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$ . . . (Art. 45)

24. $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$$= \frac{1-\tan^2 x}{1+\tan^2 x}.$$

25. $\dfrac{1-\cos 2x}{1+\cos 2x} = \dfrac{2\sin^2 x}{2\cos^2 x} = \tan^2 x.$

26. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}.$

27. $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}.$

28. $\sin 3x = 3\sin x - 4\sin^3 x$ . . . . . . (Art. 46)

29. $\cos 3x = 4\cos^3 x - 3\cos x.$

30. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$

31. $\sin^2 \dfrac{x}{2} = \dfrac{1-\cos x}{2}$ . . . . (Art. 47)

32. $\cos^2 \dfrac{x}{2} = \dfrac{1+\cos x}{2}.$

33. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$ . . . . (Art. 50)

EXAMPLES.

1. If $\sin\alpha=\frac{1}{3}$, and $\sin\beta=\frac{2}{3}$, find a value for $\sin(\alpha+\beta)$, and $\sin(\alpha-\beta)$. *Ans.* $\frac{\sqrt{5}+4\sqrt{2}}{9}$; $\frac{\sqrt{5}-4\sqrt{2}}{9}$.

2. If $\cos\alpha=\frac{4}{5}$, and $\cos\beta=\frac{40}{41}$, find a value for $\sin(\alpha+\beta)$, and $\cos(\alpha+\beta)$. *Ans.* $\frac{156}{205}$, $\frac{133}{205}$.

3. If $\cos\alpha=\frac{3}{4}$, and $\cos\beta=\frac{2}{5}$, find a value for $\sin(\alpha+\beta)$, and $\sin(\alpha-\beta)$. *Ans.* $\frac{2\sqrt{7}+3\sqrt{21}}{20}$, $\frac{2\sqrt{7}-3\sqrt{21}}{20}$.

4. If $\sin\alpha=\frac{4}{5}$, and $\sin\beta=\frac{3}{5}$, find a value for $\sin(\alpha+\beta)$, and $\cos(\alpha-\beta)$. *Ans.* 1, $\frac{24}{25}$.

5. Prove $\cos\theta+\cos 3\theta=2\cos 2\theta\cos\theta$.

6. " $2\cos\alpha\cos\beta=\cos(\alpha-\beta)+\cos(\alpha+\beta)$.

7. " $2\sin 3\theta\cos 5\theta=\sin 8\theta-\sin 2\theta$.

8. " $2\cos\frac{3}{2}\theta\cos\frac{\theta}{2}=\cos\theta+\cos 2\theta$.

9. " $\sin 4\theta\sin\theta=\frac{1}{2}(\cos 3\theta-\cos 5\theta)$.

10. " $2\cos 10^\circ\sin 50^\circ=\sin 60^\circ+\sin 40^\circ$.

Prove the following statements:

11. $\cos 3\alpha-\cos 7\alpha=2\sin 5\alpha\sin 2\alpha$.

12. $\sin 60^\circ+\sin 20^\circ=2\sin 40^\circ\cos 20^\circ$.

13. $\sin 3\theta+\sin 5\theta=2\sin 4\theta\cos\theta$.

14. $\sin 7\theta-\sin 5\theta=2\cos 6\theta\sin\theta$.

15. $\cos 5\theta + \cos 9\theta = 2\cos 7\theta \cos 2\theta.$

16. $\dfrac{\sin 2\theta + \sin\theta}{\cos\theta + \cos 2\theta} = \tan\dfrac{3\theta}{2}.$

17. $\cos(60° + A) + \cos(60° - A) = \cos A.$

18. $\cos(45° + A) + \cos(45° - A) = \sqrt{2}\cos A.$

19. $\sin(45° + A) - \sin(45° - A) = \sqrt{2}\sin A.$

20. $\cos 2\theta + \cos 4\theta = 2\cos 3\theta \cos\theta.$

21. $\cos 4\theta - \cos 6\theta = 2\sin 5\theta \sin\theta.$

22. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4\cos\theta \cos 2\theta \cos 4\theta.$

23. $\sin(A - 45°) = \dfrac{\sin A - \cos A}{\sqrt{2}}.$

24. $\sqrt{2}\sin(A + 45°) = \sin A + \cos A.$

25. $\cos(A + 45°) + \sin(A - 45°) = 0.$

26. $\dfrac{\tan(\theta - \phi) + \tan\phi}{1 - \tan(\theta - \phi)\tan\phi} = \tan\theta.$

27. $\dfrac{\tan(\theta + \phi) - \tan\theta}{1 + \tan(\theta + \phi)\tan\theta} = \tan\phi.$

28. $\cos(\theta + \phi) - \sin(\theta - \phi) = 2\sin\left(\dfrac{\pi}{4} - \theta\right)\cos\left(\dfrac{\pi}{4} - \phi\right).$

29. $\sin n\theta \cos\theta + \cos n\theta \sin\theta = \sin(n + 1)\theta.$

30. $\cot\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{\cot\theta + 1}{1 - \cot\theta}.$

31. $\tan\left(\theta - \dfrac{\pi}{4}\right) + \cot\left(\theta + \dfrac{\pi}{4}\right) = 0.$

32. $\cot\left(\theta - \dfrac{\pi}{4}\right) + \tan\left(\theta + \dfrac{\pi}{4}\right) = 0.$

33. If $\tan x = 1$, and $\tan y = \dfrac{1}{\sqrt{3}}$, prove that

$$\tan(x+y) = 2 + \sqrt{3}.$$

34. If $\tan\alpha = \dfrac{m}{m+1}$, and $\tan\beta = \dfrac{1}{2m+1}$, prove that

$$\tan(\alpha+\beta) = 1.$$

35. If $\tan\alpha = m$, and $\tan\beta = n$, prove that

$$\cos(\alpha+\beta) = \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

36. If $\tan\theta = (a+1)$, and $\tan\phi = (a-1)$, prove that

$$2\cot(\theta-\phi) = a^2.$$

Prove the following statements:

37. $\dfrac{2-\sec^2\theta}{\sec^2\theta} = \cos 2\theta.$

38. $\cos^2\theta\,(1-\tan^2\theta) = \cos 2\theta.$

39. $\cot 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}.$

40. $\sec 2\theta = \dfrac{\cot^2\theta + 1}{\cot^2\theta - 1}.$

41. $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$

42. $\left(\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right)^2 = 1 - \sin\theta.$

43. $\dfrac{1+\sec\theta}{\sec\theta} = 2\cos^2\dfrac{\theta}{2}.$

44. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \dfrac{1-\tan\theta}{1+\tan\theta}.$

45. $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\tan\dfrac{\theta}{2}}{1-\tan\dfrac{\theta}{2}}.$

46. $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2.$

47. $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta.$

48. $\dfrac{\sin 4\theta}{\sin 2\theta} = 2\cos 2\theta.$

49. $\tan(45^\circ + x) - \tan(45^\circ - x) = 2\tan 2x.$

50. $\tan(45^\circ - x) + \cot(45^\circ - x) = 2\sec 2x.$

51. $\dfrac{\cos 3x}{\cos x} = 2\cos 2x - 1.$

52. If $\tan\theta = \frac{1}{7}$, and $\tan\phi = \frac{2}{11}$, prove $\tan(2\theta + \phi) = \frac{1}{2}$.

53. Prove that $\tan\frac{\theta}{2}$ and $\cot\frac{\theta}{2}$ are the roots of the equation

$$x^2 - 2x\operatorname{cosec}\theta + 1 = 0.$$

54. If $\tan\theta = \frac{b}{a}$, prove that

$$\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \frac{2\cos\theta}{\sqrt{\cos 2\theta}}.$$

Prove the following statements when we take for $\sin^{-1}$, $\cos^{-1}$, etc., their least *positive* value.

55. $\sin^{-1}\frac{1}{2} = \cos^{-1}\frac{\sqrt{3}}{2} = \cot^{-1}\sqrt{3}.$

56. $2\tan^{-1}(\cos 2\theta) = \tan^{-1}\left(\dfrac{\cot^2\theta - \tan^2\theta}{2}\right).$

57. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}.$

58. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{13}{85} = \frac{\pi}{2}.$

59. $\tan^{-1}\sqrt{5}\,(2-\sqrt{3})-\cot^{-1}\sqrt{5}\,(2+\sqrt{3})=\cot^{-1}\sqrt{5}.$

60. $\sec^{-1}3=2\cot^{-1}\sqrt{2}.$

61. $2\cot^{-1}x=\operatorname{cosec}^{-1}\dfrac{1+x^2}{2x}.$

62. $\tan^{-1}\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\tan^{-1}\sqrt{\dfrac{3}{2}}=\dfrac{3\pi}{4}.$

63. $\sin^{-1}\dfrac{1}{\sqrt{5}}+\cot^{-1}3=\dfrac{\pi}{4}.$

64. $\cos^{-1}\dfrac{63}{65}+2\tan^{-1}\dfrac{1}{5}=\sin^{-1}\dfrac{3}{5}.$

65. If $\theta=\sin^{-1}\dfrac{3}{5}$, and $\phi=\cos^{-1}\dfrac{3}{5}$, then $\theta+\phi=90^\circ$.

66. Prove that $\cos\,(2\tan^{-1}x)=\dfrac{1-x^2}{1+x^2}.$

67. “ “ $\tan^{-1}\dfrac{1}{2}+\operatorname{cosec}^{-1}\sqrt{10}=\dfrac{\pi}{4}.$

68. “ “ $2\tan^{-1}\dfrac{2}{3}-\operatorname{cosec}^{-1}\dfrac{5}{3}=\sin^{-1}\dfrac{33}{65}.$

69. “ “ $2\tan^{-1}\dfrac{1}{2}+\cos^{-1}\dfrac{4}{5}=\dfrac{\pi}{2}.$

70. “ “ $\sin^{-1}(\cos x)+\cos^{-1}(\sin y)+x+y=\pi.$

71. “ “ $\tan^{-1}\dfrac{1}{1+x}+\tan^{-1}\dfrac{1}{1-x}+\tan^{-1}\dfrac{2}{x^2}=n\pi.$

72. “ “ $\tan^{-1}\dfrac{x-1}{x}+\tan^{-1}\dfrac{1}{2x-1}=n\pi+\dfrac{\pi}{4}.$

73. “ “ $\sin^{-1}x-\sin^{-1}y$

$$=\cos^{-1}\left(xy\pm\sqrt{1-x^2-y^2+x^2y^2}\right).$$

Solve the following equations:

74. $\tan\theta + \cot\theta = 2.$ *Ans.* 45°.

75. $2\sin^2\theta + \sqrt{2}\cos\theta = 2.$ 90°, or 45°.

76. $3\tan^2\theta - 4\sin^2\theta = 1.$ 45°.

77. $2\sin^2\theta + \sqrt{2}\sin\theta = 2.$ 45°.

78. $\cos^2\theta - \sqrt{3}\cos\theta + \frac{3}{4} = 0.$ 30°.

79. $\sin 5\theta = 16\sin^5\theta.$ $\pm\frac{\pi}{6}.$

80. $\sin 9\theta - \sin\theta = \sin 4\theta.$ $\pm\frac{\pi}{15}.$

81. $2\sin\theta = \tan\theta.$ $\pm\frac{\pi}{3}.$

82. $6\cot^2\theta - 4\cos^2\theta = 1.$ $\pm\frac{\pi}{3}.$

83. $\tan\theta + \tan(\theta - 45^\circ) = 2.$ $\pm\frac{\pi}{3}.$

84. $\cos\theta + \sqrt{3}\sin\theta = 2.$ $\pm\frac{\pi}{3}.$

85. $\tan(\theta + 45^\circ) = 1 + \sin 2\theta.$ $-\frac{\pi}{4}.$

86. $(\cot\theta - \tan\theta)^2(2 + \sqrt{3}) = 4(2 - \sqrt{3}).$ $\pm\frac{5\pi}{24}.$

87. $\operatorname{cosec}\theta\cot\theta = 2\sqrt{3}.$ $\pm\frac{\pi}{6}.$

88. $\operatorname{cosec}\theta + \cot\theta = \sqrt{3}.$ $+\frac{1}{3}\pi.$

89. $\sin\frac{\theta}{2} = \operatorname{cosec}\theta - \cot\theta.$ 0°.

90. $\sin^2\theta + \cos^2 2\theta = \frac{3}{4}.$ $\pm\frac{\pi}{10}$, or $\pm\frac{3}{10}\pi.$

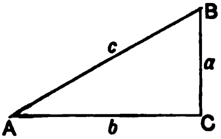
CHAPTER IV.

RELATIONS BETWEEN THE SIDES OF A TRIANGLE AND THE FUNCTIONS OF ITS ANGLES.

53. Formulæ. — In this chapter we shall deduce formulæ which express certain relations between the sides of a triangle and the functions of its angles. These relations will be applied in the next chapter to the *solution of triangles.* One of the principal objects of Trigonometry, as its name implies (Art. 1), is to establish certain relations between the sides and angles of triangles, so that when some of these are known the rest may be determined.

RIGHT TRIANGLES.

54. Let ABC be a triangle, right-angled at C. Denote the angles of the triangle by the letters A, B, C, and the lengths of the sides respectively opposite these angles, by the letters a, b, c.* Then we have (Art. 14) the following relations:



$$a = c \sin A = c \cos B = b \tan A = b \cot B \quad . \quad . \quad (1)$$

$$b = c \sin B = c \cos A = a \tan B = a \cot A \quad . \quad . \quad (2)$$

$$c = b \sec A = a \sec B = b \operatorname{cosec} B = a \operatorname{cosec} A \quad . \quad (3)$$

which may be expressed in the following general theorems:

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* The student must remember that a, b, c, are *numbers* expressing the lengths of the sides in terms of some unit of length, such as a foot or a mile. The unit may be whatever we please, but must be the same for all the sides.

I. *In a right triangle each side is equal to the product of the hypotenuse into the sine of the opposite angle or the cosine of the adjacent angle.*

II. *In a right triangle each side is equal to the product of the other side into the tangent of the angle adjacent to that other side, or the cotangent of the angle adjacent to itself.*

III. *In a right triangle the hypotenuse is equal to the product of a side into the secant of its adjacent angle, or the cosecant of its opposite angle.*

EXAMPLES.

In a right triangle ABC, in which C is a right angle, prove the following:

1. $\tan B = \cot A + \cos C.$
2. $\sin 2A = \sin 2B.$
3. $\cos 2A + \cos 2B = 0.$
4. $\sin 2A = \frac{2ab}{c^2}.$
5. $\operatorname{cosec} 2B = \frac{a}{2b} + \frac{b}{2a}.$
6. $\cos 2A = \frac{b^2 - a^2}{c^2}.$
7. $\tan 2A = \frac{2ab}{b^2 - a^2}.$
8. $\sin 3A = \frac{3ab^2 - a^3}{c^3}.$

OBLIQUE TRIANGLES.

55. Law of Sines.—*In any triangle the sides are proportional to the sines of the opposite angles.*

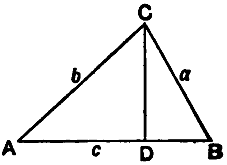
Let ABC be any triangle. Draw CD perpendicular to AB.

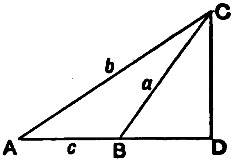
We have, then, in both figures

$$CD = a \sin B = b \sin A. \quad \text{(Art. 54)}$$

$$\therefore\ a \sin B = b \sin A.$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B}.$$





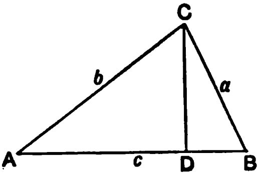
Similarly, by drawing a perpendicular from A or B to the opposite side, we may prove that

$$\frac{b}{\sin B}=\frac{c}{\sin C}, \text{ and } \frac{c}{\sin C}=\frac{a}{\sin A}.$$

$$\therefore \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

or $$a : b : c = \sin A : \sin B : \sin C.$$

56. Law of Cosines. — *In any triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these sides and the cosine of the included angle.*



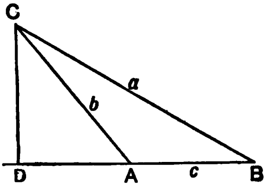
In an *acute-angled* triangle (see first figure) we have (Geom., Book III., Prop. 26)

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \times AD,$$

or $$a^2 = b^2 + c^2 - 2c \cdot AD.$$

But $$AD = b \cos A.$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A.$$



In an *obtuse-angled* triangle (see second figure) we have (Geom., Book III., Prop. 27)

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 + 2\,AB \times AD,$$

or $$a^2 = b^2 + c^2 + 2c \cdot AD.$$

But $$AD = b \cos CAD = -b \cos A.$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly, $$b^2 = c^2 + a^2 - 2ca \cos B,$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

NOTE. — When one equation in the solution of triangles has been obtained, the other two may generally be obtained by advancing the letters so that a becomes b, b becomes c, and c becomes a; the order is abc, bca, cab. It is obvious that the formulæ thus obtained are true, since the naming of the sides makes no difference, provided the right order is maintained.

57. Law of Tangents. — *In any triangle the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

By Art. 55, $a : b = \sin A : \sin B$.

By composition and division,

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \text{ by (1) of Art. 52} \quad (1)$$

Similarly,
$$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)} \quad \ldots\ldots\ldots \quad (2)$$

$$\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)} \quad \ldots\ldots\ldots \quad (3)$$

Since $\tan \frac{1}{2}(A+B) = \tan(90° - \frac{1}{2}C) = \cot \frac{1}{2}C$,

the result in (1) may be written

$$\frac{a+b}{a-b} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)} \quad \ldots\ldots\ldots \quad (4)$$

and similar expressions for (2) and (3).

58. To show that in any triangle $c = a \cos B + b \cos A$.

In an *acute-angled* triangle (first figure of Art. 56) we have

$$c = DB + DA$$

$$= a \cos B + b \cos A.$$

In an *obtuse-angled* triangle (second figure of Art. 56) we have

$$c = DB - DA$$

$$= a \cos B - b \cos CAD.$$

$$\therefore c = a \cos B + b \cos A.$$

Similarly,
$$b = c \cos A + a \cos C,$$

$$a = b \cos C + c \cos B.$$

EXAMPLES.

1. In the triangle ABC prove (1)

$$a+b:c=\cos\tfrac{1}{2}(A-B):\sin\tfrac{1}{2}C,$$

and (2) $$a-b:c=\sin\tfrac{1}{2}(A-B):\cos\tfrac{1}{2}C.$$

2. If AD bisects the angle A of the triangle ABC, prove

$$BD:DC=\sin C:\sin B.$$

3. If AD′ bisects the external vertical angle A, prove

$$BD':CD'=\sin C:\sin B.$$

4. Hence prove $$\frac{1}{DC}=\frac{2\cos\frac{1}{2}A\cos\frac{1}{2}(B-C)}{a\sin B};$$

and also $$\frac{1}{D'C}=\frac{2\sin\frac{1}{2}A\sin\frac{1}{2}(C-B)}{a\sin B}.$$

59. To express the Sine, the Cosine, and the Tangent of Half an Angle of a Triangle in Terms of the Sides.

I. By Art. 56 we have

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=1-2\sin^2\frac{A}{2} \quad . \quad . \quad . \quad . \quad \text{(Art. 45)}$$

$$\therefore\ 2\sin^2\frac{A}{2}=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b-c)(a-b+c)}{2bc}.$$

Let $a+b+c=2s$;

then $a+b-c=2(s-c)$, and $a-b+c=2(s-b)$.

$$\therefore\ 2\sin^2\frac{A}{2}=\frac{2(s-c)2(s-b)}{2bc}.$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}} \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, $$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ac}} \quad \ldots\ldots \quad (2)$$

$$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}} \quad \ldots\ldots \quad (3)$$

II. $$\cos A = 2\cos^2\frac{A}{2}-1. \quad \ldots\ldots \quad \text{(Art. 45)}$$

$$\therefore\ 2\cos^2\frac{A}{2}=1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(a+b+c)(b+c-a)}{2bc}$$

$$=\frac{2s\cdot 2(s-a)}{2bc}.$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}} \quad \ldots\ldots \quad (4)$$

Similarly, $$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}} \quad \ldots\ldots \quad (5)$$

$$\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}} \quad \ldots\ldots \quad (6)$$

III. Dividing (1) by (4), we get

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \ldots\ldots \quad (7)$$

Similarly, $$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \quad \ldots\ldots \quad (8)$$

$$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \quad \ldots\ldots \quad (9)$$

Since any angle of a triangle is $< 180°$, the half angle is $< 90°$; therefore the *positive* sign must be given to the radicals which occur in this article.

60. To express the Sine of an Angle in Terms of the Sides.

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(Art. 45)}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}. \qquad \text{(Art. 59)}$$

$$\therefore \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly, $\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)},$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

Cor. $\sin A = \frac{1}{2bc}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4},$

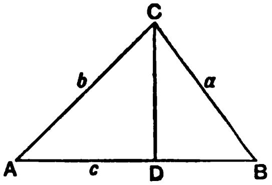
and similar expressions for sin B, sin C.

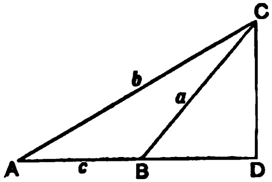
EXAMPLES.

In any triangle ABC prove the following statements:

1. $a(b\cos C - c\cos B) = b^2 - c^2.$
2. $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a + b + c.$
3. $\frac{\sin A + 2\sin B}{a + 2b} = \frac{\sin C}{c}.$
4. $\frac{\sin^2 A - m\sin^2 B}{a^2 - mb^2} = \frac{\sin^2 C}{c^2}.$
5. $a\cos A + b\cos B - c\cos C = 2c\cos A\cos B.$
6. $\frac{\cos A}{\sin B\sin C} + \frac{\cos B}{\sin C\sin A} + \frac{\cos C}{\sin A\sin B} = 2.$
7. $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0.$
8. $\tan\frac{1}{2}A\tan\frac{1}{2}B = \frac{s-c}{s}.$
9. $\tan\frac{1}{2}A + \tan\frac{1}{2}B = (s-b) \div (s-c).$

61. Expressions for the Area of a Triangle.





(1) *Given two sides and their included angle.*

Let S denote the area of the triangle ABC. Then by Geometry,

$$2S = c \times CD.$$

But in either figure, by Art. 54,

$$CD = b \sin A.$$

$$\therefore\ S = \tfrac{1}{2} bc \sin A.$$

Similarly, $S = \tfrac{1}{2} ac \sin B,$

$$S = \tfrac{1}{2} ab \sin C.$$

(2) *Given one side and the angles.*

Since $a : b = \sin A : \sin B$ . . . . . . (Art. 55)

$$\therefore\ b = \frac{a \sin B}{\sin A},$$

which in $S = \tfrac{1}{2} ab \sin C$, gives

$$S = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

Similarly, $S = \dfrac{b^2 \sin A \sin C}{2 \sin B} = \dfrac{c^2 \sin A \sin B}{2 \sin C}.$

(3) *Given the three sides.*

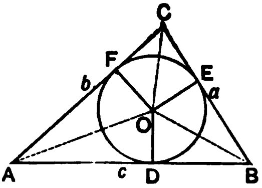
$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \qquad \text{(Art. 60)}$$

Substituting in

$$S = \tfrac{1}{2} bc \sin A,$$

we get $S = \sqrt{s(s-a)(s-b)(s-c)}.$

62. Inscribed Circle. — *To find the radius of the inscribed circle of a triangle.*



Let ABC be a triangle, O the centre of the inscribed circle, and r its radius. Draw radii to the points of contact D, E, F; and join OA, OB, OC. Then

$$S = \text{area of } ABC$$

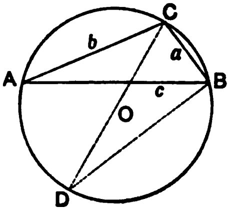
$$= \triangle AOB + \triangle BOC + \triangle COA$$

$$= \tfrac{1}{2} rc + \tfrac{1}{2} ra + \tfrac{1}{2} rb$$

$$= r\frac{a+b+c}{2} = rs \quad . \quad . \quad . \quad . \quad \text{(Art. 59)}$$

$$\therefore\ r = \frac{S}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad . \quad \text{(Art. 61)}$$

63. Circumscribed Circle. — *To find the radius of the circumscribed circle of a triangle in terms of the sides of the triangle.*



Let O be the centre of the circle described about the triangle ABC, and R its radius.

Through O draw the diameter CD and join BD.

Then $\angle BDC = \angle BAC = \angle A.$

$$\therefore\ BC = 2R \sin A, \text{ or } a = 2R\sin A.$$

$$\therefore\ R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} \quad . \quad . \quad . \quad (1)$$

But $\sin A = \dfrac{2S}{bc}$ . . . . (Art. 61)

$$\therefore\ R = \frac{abc}{4S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLES.

In a right triangle ABC, in which C is the right angle, prove the following:

1. $\cos 2B = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B}$.

2. $\sin^2 \dfrac{B}{2} = \dfrac{c - a}{2c}$.

3. $\left(\cos \dfrac{A}{2} + \sin \dfrac{A}{2}\right)^2 = \dfrac{a + c}{c}$.

4. $\cos^2 \dfrac{A}{2} = \dfrac{b + c}{2c}$.

5. $\sin (A - B) + \cos 2A = 0$.

6. $\dfrac{a - b}{a + b} = \tan \dfrac{A - B}{2}$.

7. $\sin (A - B) + \sin (2A + C) = 0$.

8. $\tan \frac{1}{2} A = \dfrac{a}{b + c}$.

9. $(\sin A - \sin B)^2 + (\cos A + \cos B)^2 = 2$.

In any triangle ABC, prove the following statements:

10. $(a + b) \sin \dfrac{C}{2} = c \cos \dfrac{A - B}{2}$.

11. $(b - c) \cos \dfrac{A}{2} = a \sin \dfrac{B - C}{2}$.

12. $a(b^2 + c^2) \cos A + b(c^2 + a^2) \cos B + c(a^2 + b^2) \cos C = 3abc$.

13. $\dfrac{a - b}{c} = \dfrac{\cos B - \cos A}{1 + \cos C}$.

14. $\dfrac{b + c}{a} = \dfrac{\cos B + \cos C}{1 - \cos A}$.

15. $\sqrt{bc\sin B\sin C}=\dfrac{b^2\sin C+c^2\sin B}{b+c}.$

16. $a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C.$

17. $b+c-a=(b+c)\cos A-(c-a)\cos B+(a-b)\cos C.$

18. $a\cos(A+B+C)-b\cos(B+A)-c\cos(A+C)=0.$

19. $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}.$

20. $\tan A=\dfrac{a\sin C}{b-a\cos C}.$

21. $b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}=s.$

22. $\tan\dfrac{B}{2}\tan\dfrac{C}{2}=\dfrac{b+c-a}{b+c+a}.$

23. $\tan\dfrac{A}{2}(b+c-a)=\tan\dfrac{B}{2}(c+a-b).$

24. $c^2=(a+b)^2\sin^2\dfrac{C}{2}+(a-b)^2\cos^2\dfrac{C}{2}.$

25. $c(\cos A+\cos B)=2(a+b)\sin^2\dfrac{C}{2}.$

26. $c(\cos A-\cos B)=2(b-a)\cos^2\dfrac{C}{2}.$

27. $\tan B\div\tan C=(a^2+b^2-c^2)\div(a^2-b^2+c^2).$

28. $a^2+b^2+c^2=2(ab\cos C+bc\cos A+ca\cos B).$

29. $\cos^2\dfrac{A}{2}\div\cos^2\dfrac{B}{2}=(s-a)\div b(s-b).$

30. $b\sin^2\dfrac{C}{2}+c\sin^2\dfrac{B}{2}=s-a.$

31. If p is the length of the perpendicular from A on BC,

$$\sin A=\frac{ap}{bc}.$$

32. If $\sqrt{bc \sin B \sin C} = \dfrac{b^2 \sin B + c^2 \sin C}{b + c}$, then $B = C$.

33. $a \cos\dfrac{B}{2} \cos\dfrac{C}{2} \operatorname{cosec}\dfrac{A}{2} = s$.

34. If $\cos A = \frac{3}{5}$, and $\cos B = \frac{12}{13}$, then $\cos C = -\frac{16}{65}$.

35. If $\sin^2 B + \sin^2 C = \sin^2 A$, then $A = 90°$.

36. If D is the middle point of BC, prove that

$$4\,\overline{AD}^2 = 2b^2 + 2c^2 - a^2.$$

37. If $a = 2b$, and $A = 3B$, prove that $C = 60°$.

38. If D, E, F, are the middle points of the sides, BC, CA, AB, prove

$$4(\overline{AD}^2 + \overline{BE}^2 + \overline{CF}^2) = 3(a^2 + b^2 + c^2).$$

39. If $\cos B = \dfrac{\sin A}{2 \sin C}$, prove that $B = C$.

40. If $a^2 = b^2 - bc + c^2$, prove that $A = 60°$.

41. If the sides of a triangle are a, b, and $\sqrt{a^2 + ab + b^2}$, prove that its greatest angle is 120°.

42. Prove that the vertical angle of any triangle is divided by the median which bisects the base, into segments whose sines are inversely proportional to the adjacent sides.

43. Find the area of the triangle ABC when $a = 625$, $b = 505$, $c = 904$ yards. *Ans.* 151872 sq. yards.

44. Prove that the lengths of the sides of the pedal triangle, that is, the triangle formed by joining the feet of the perpendiculars, are $a \cos A$, $b \cos B$, $c \cos C$, respectively.

45. Prove that the angles of the pedal triangle are, respectively, $\pi - 2A$, $\pi - 2B$, $\pi - 2C$.

46. Prove that the area of the incircle : area of the triangle $:: \pi : \cot\dfrac{A}{2} \cot\dfrac{B}{2} \cot\dfrac{C}{2}$.

CHAPTER V.

SOLUTION OF TRIANGLES.

64. Triangles. — In every triangle there are *six elements*, the three sides and the three angles. When any three elements are given, one at least of the three being a side, the other three can be calculated. The process of determining the unknown elements from the known is called the *solution of triangles.*

Note. — If the three *angles* only of a triangle are given, it is impossible to determine the sides, for there is an infinite number of triangles that are equiangular to one another.

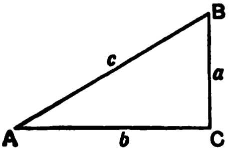
Triangles are divided in Trigonometry into *right* and *oblique*. We shall commence with right triangles, and shall suppose C the right angle.

RIGHT TRIANGLES.

65. There are Four Cases of Right Triangles.

I. *Given one side and the hypotenuse.*
II. *Given an acute angle and the hypotenuse.*
III. *Given one side and an acute angle.*
IV. *Given the two sides.*

Let ABC be a triangle, right-angled at C, and let a, b, and c, as before, be the sides opposite the angles A, B, and C, respectively.



The formulæ for the solution of right triangles are (1), (2), (3) of Art. 54.

Logarithmic Solution.

$$
\begin{array}{r@{\;=\;}l|r@{\;=\;}l}
\multicolumn{2}{c|}{\log a = \log c + \log \sin A.} & \multicolumn{2}{c}{\log b = \log c + \log \cos A.} \\
\log c & 2.09691 & \log c & 2.09691 \\
\log \sin A & \underline{9.91051} & \log \cos A & \underline{9.76431} \\
\log a^{*} & 2.00742 & \log b^{*} & 1.86122 \\
\therefore\ a & 101.72. & \therefore\ b & 72.647.
\end{array}
$$

Ex. 2. Given $A = 37^\circ 10'$, $c = 8762$; find a and b.

Ans. 5293.4; 6982.3.

68. Case III. — *Given a side and an acute angle, as A and a; to find B, b, c.*

We have $B = 90^\circ - A.$

Also $b = \dfrac{a}{\tan A}$, and $c = \dfrac{a}{\sin A}.$

$$\therefore \log b = \log a - \log \tan A,$$

and $$\log c = \log a - \log \sin A.$$

Ex. 1. Given $A = 32^\circ 15' 24''$, $a = 5472.5$; find B, b, c.

Solution.

$$B = 90^\circ - A = 57^\circ 44' 36''.$$

$$
\begin{array}{r@{\;=\;}l|r@{\;=\;}l}
\multicolumn{2}{c|}{\log b = \log a - \log \tan A.} & \multicolumn{2}{c}{\log c = \log a - \log \sin A.} \\
\log a & 3.73818 & \log a & 3.73818 \\
\log \tan A & \underline{9.80011} & \log \sin A & \underline{9.72731} \\
\log b & 3.93807 & \log c & 4.01087 \\
\therefore\ b & 8671.0. & \therefore\ c & 10253.5.
\end{array}
$$

Ex. 2. Given $A = 34^\circ 18'$, $a = 237.6$; find B, b, c.

Ans. $B = 55^\circ 42'$; $b = 348.31$; $c = 421.63$.

---

[*] *Ten* is *rejected* because the tabular logarithmic functions are too large by ten.

69. Case IV. — *Given the two sides, as a and b; to find A, B, c.*

We have $$\tan A = \frac{a}{b};\ \text{then}\ B = 90^\circ - A.$$

Also $$c = a \operatorname{cosec} A = \frac{a}{\sin A}.$$

$$\therefore\ \log \tan A = \log a - \log b,$$

and $$\log c = \log a - \log \sin A.$$

Ex. Given $a = 2266.35$, $b = 5439.24$; find A, B, c.

Solution.

$\log \tan A = \log a - \log b.$	$\log c = \log a - \log \sin A.$
$\log a = 3.35533$	$\log a = 3.35533$
$\log b = 3.73554$	$\log \sin A = 9.58503$
$\log \tan A = 9.61979$	$\log c = 3.77030$
$\therefore\ A = 22^\circ\, 37'\, 11''.$	$\therefore\ c = 5892.5.$
$\therefore\ B = 67^\circ\, 22'\, 49''.$	

Note. — In this example we might have found c by means of the formula $c = \sqrt{a^2 + b^2}$; but we would have had to go through the process of *squaring* the values of a and b. If these values are simple numbers, it is often easier to find c in this way; but this value of c is not adapted to logarithms. A formula which consists entirely of *factors* is always preferred to one which consists of *terms*, when any of those terms contain any power of the quantities involved.

70. When a Side and the Hypotenuse are nearly Equal. — When a side and the hypotenuse are given, as a and c in Case I., and are nearly equal in value, the angle A is very near 90°, and cannot be determined with much accuracy from the tables, because the sines of angles near 90° differ very little from one another. It is therefore desirable, in this case, to find B first, by either of the following formulæ:

$$\sin \frac{B}{2} = \sqrt{\frac{1 - \cos B}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 46)}$$

$$= \sqrt{\frac{c - a}{2c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tan\frac{B}{2} = \sqrt{\frac{1-\cos B}{1+\cos B}} \quad \cdots\cdots \quad \text{(Art. 46)}$$

$$= \sqrt{\frac{c-a}{c+a}} \quad \cdots\cdots\cdots \quad (2)$$

Then $$b = c\cos A \quad \cdots\cdots\cdots \quad (3)$$

or $$= \sqrt{(c+a)(c-a)} \quad \cdots\cdots \quad (4)$$

Ex. 1. Given $a = 4602.21059$, $c = 4602.836$; find B.

$$\begin{array}{lr} c - a = 0.62541, & \log(c-a) = \bar{1}.79616 \\ 2c = 9205.672, & \log 2c = 3.96405 \\ & 2\underline{)\overline{\bar{5}.83211}} \\ & \log\sin\frac{B}{2} = 7.91606 \\ B = 56'\,40''.32. & \therefore\ \frac{B}{2} = 28'\,20''.16. \end{array}$$

Note.—The characteristic $\bar{5}$ is increased numerically to $\bar{6}$ to make it divisible by 2. Ten is then added to the characteristic $\bar{3}$, making it 7, so as to agree with the tables.

There is a slight error in the above value of B on account of the irregular differences of the log sines for angles near 0°. A more accurate value may be found by the principle that the sines of small angles are approximately proportional to the angles (Treatise on Trigonometry, Art. 130).

EXAMPLES.

The following right triangles must be solved by logarithms.

1. Given $a = 60$, $c = 100$; find A, B, b.

Ans. $A = 36°\,52'$; $B = 53°\,8'$; $b = 80$.

2. Given $a = 137.66$, $c = 240$; find A, B, b.

Ans. $A = 35°$; $B = 55°$; $b = 196.59$.

3. Given $a = 147$, $c = 184$; find A, B, b.
Ans. $A = 53° 1' 35''$; $B = 36° 58' 25''$; $b = 110.67$.

4. Given $a = 100$, $c = 200$; find A, B, b.
Ans. $A = 30°$; $B = 60°$; $b = 100\sqrt{3}$.

5. Given $A = 40°$, $c = 100$; find B, a, b.
Ans. $B = 50°$; $a = 64.279$; $b = 76.604$.

6. Given $A = 30°$, $c = 150$; find B, a, b.
Ans. $B = 60°$; $a = 75$; $b = 75\sqrt{3}$.

7. Given $A = 32°$, $c = 1760$; find B, a, b.
Ans. $B = 58°$; $a = 932.66$; $b = 1492.57$.

8. Given $A = 35° 16' 25''$, $c = 672.3412$; find B, a, b.
Ans. $B = 54° 43' 35''$; $a = 388.26$; $b = 548.9$.

9. Given $A = 75°$, $a = 80$; find B, b, c.
Ans. $B = 15°$; $b = 80(2 - \sqrt{3})$; $c = 80(\sqrt{6} - \sqrt{2})$.

10. Given $A = 36°$, $a = 520$; find B, b, c.
Ans. $B = 54°$; $b = 715.72$; $c = 884.68$.

11. Given $A = 34° 15'$, $a = 843.2$; find B, b, c.
Ans. $B = 55° 45'$; $c = 1498.2$.

12. Given $A = 67° 37' 15''$, $b = 254.73$; find B, a, c.
Ans. $B = 22° 22' 45''$; $a = 618.66$; $c = 669.05$.

13. Given $a = 75$, $b = 75$; find A, B, c.
Ans. $A = 45° = B$; $c = 75\sqrt{2}$.

14. Given $a = 21$, $b = 20$; find A, B, c.
Ans. $A = 46° 23' 50''$; $c = 29$.

15. Given $a = 300.43$, $b = 500$; find A, B, c.
Ans. $A = 31°$; $B = 59°$; $c = 583.31$.

16. Given $a = 4845$, $b = 4742$; find A, B, c.
Ans. $A = 45° 36' 56''$.

OBLIQUE TRIANGLES.

71. There are Four Cases of Oblique Triangles.

I. *Given a side and two angles.*
II. *Given two sides and the angle opposite one of them.*
III. *Given two sides and the included angle.*
IV. *Given the three sides.*

The formulæ for the solution of oblique triangles will be taken from Chap. IV. Special attention must be given to the following three, proved in Arts. 55, 56, 57.

(1) The Sine-rule, $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

(2) The Cosine-rule, $$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

(3) The Tangent-rule, $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

72. Case I. — *Given a side and two angles, as a, B, C; find A, b, c.*

(1) $A=180^\circ-(B+C)$. $\therefore$ A is found.

(2) $\dfrac{b}{\sin B}=\dfrac{a}{\sin A}$. $\therefore\ b=\dfrac{a\sin B}{\sin A}$.

(3) $\dfrac{c}{\sin C}=\dfrac{a}{\sin A}$. $\therefore\ c=\dfrac{a\sin C}{\sin A}$.

These determine b and c.

Ex. 1. Given $a=7012.6$, $B=38^\circ\,12'\,48''$, $C=60^\circ$; find A, b, c.

Solution.

$$A=180^\circ-(B+C)=81^\circ\,47'\,12''.$$

$$\begin{array}{r|r}
\log a = 3.84587 & \log a = 3.84587 \\
\log \sin B = 9.79140 & \log \sin C = 9.93753 \\
\text{colog}^{*}\sin A = \underline{0.00448} & \text{colog}\sin A = \underline{0.00448} \\
\log b = 3.64175 & \log c = 3.78788 \\
\therefore\ b = 4382.8. & \therefore\ c = 6135.9.
\end{array}$$

---

* See Art. 69.

Ex. 2. Given $a=1000$, $B=45°$, $C=127° 19'$; find A, b, c.

Ans. $A=7° 41'$; $b=5288.8$; $c=5948.5$.

73. Case II. — *Given two sides and the angle opposite one of them, as a, b, A; find B, C, c.*

(1) $\sin B=\frac{b\sin A}{a}$; thus B is found.

(2) $C=180°-(A+B)$; thus C is found.

(3) $c=\frac{a\sin C}{\sin A}$; thus c is found.

This is usually known as the *ambiguous* case, as shown in geometry (B. II., Prop. 31). The ambiguity is found in the equation

$$\sin B=\frac{b\sin A}{a}.$$

Since the angle is determined by its sine, it admits of two values, which are supplements of each other (Art. 29). Therefore either value of B may be taken, unless excluded by the conditions of the problem.

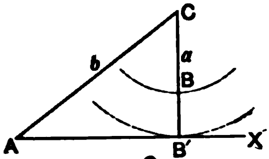
I. If $a<b\sin A$, $\sin B>1$, which is impossible; and therefore there is no triangle with the given parts.

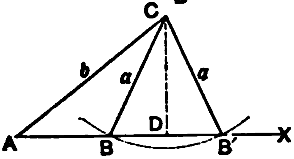
II. If $a=b\sin A$, $\sin B=1$, and $B=90°$; therefore there is one triangle — a right triangle — with the given parts.

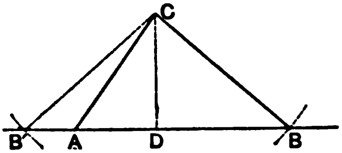
III. If $a>b\sin A$, and $<b$, $\sin B<1$; hence there are two values of B, one being the supplement of the other, *i.e.*, one acute, the other obtuse, and both are admissible; therefore there are two triangles with the given parts.

IV. If $a>b$, then $A>B$, and since A is given, B must be acute; thus there is only one triangle with the given parts.

These four cases may be illustrated geometrically.







Draw A, the given angle. Make $AC = b$; draw the perpendicular CD, which $= b \sin A$. With centre C and radius a, describe a circle.

I. If $a < b \sin A$, the circle will not meet AX, and therefore no triangle can be formed with the given parts.

II. If $a = b \sin A$, the circle touches AX in B′; therefore there is one triangle, right-angled at B.

III. If $a > b \sin A$, and $< b$, the circle cuts AX in two points B and B′, on the same side of A; thus there are two triangles ABC and AB′C, each having the given parts, the angles ABC, AB′C being supplementary.

IV. If $a > b$, the circle cuts AX on *opposite* sides of A, and only the triangle ABC has the given parts, because the angle B′AC of the triangle AB′C is not the given angle A, but its supplement.

These results may be stated as follows:

$a < b \sin A$,	no solution.
$a = b \sin A$,	one solution (right triangle).
$a > b \sin A$ and $< b$,	two solutions.
$a > b$,	one solution.

These results may be obtained algebraically thus:

We have $$a^2 = b^2 + c^2 - 2bc \cos A \quad . \quad . \quad . \quad \text{(Art. 56)}$$

$$\therefore\ c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A},$$

giving two roots, *real* and *unequal*, *equal* or *imaginary*, according as $a >, =,$ or $< b \sin A$.

A discussion of these two values of c gives the same results as are found in the above four cases. We leave the discussion as an exercise for the student.

NOTE. — When two sides and the angle opposite the greater are given, there can be no ambiguity, for the angle opposite the less must be acute.

When the given angle is a right angle or obtuse, the other two angles are both acute, and there can be no ambiguity.

In the solution of triangles there can be no ambiguity, except when an angle is determined by the sine or cosecant, and in no case whatever when the triangle has a right angle.

Ex. 1. Given $a = 7$, $b = 8$, $A = 27° 47' 45''$; find B, C, c.

Solution.

$$\begin{array}{rl|rl} \log b = & 0.90309 & \log a = & 0.84510 \\ \log \sin A = & 9.66869 & \log \sin C = & 9.93753 \\ \operatorname{colog} a = & \underline{9.15490} & \operatorname{colog} \sin A = & \underline{0.33131} \\ \log \sin B = & 9.72668 & \log c = & 1.11394 \end{array}$$

$$\therefore\ B = 32° 12' 15'', \text{ or } 147° 47' 45''. \qquad \therefore\ c = 13.$$

$$\therefore\ C = 120°, \text{ or } 4° 24' 30''.$$

Taking the second value of C as follows:

$$\begin{aligned} \log a &= 0.84510 \\ \log \sin C &= 8.88572 \\ \operatorname{colog} \sin A &= \underline{0.33131} \\ \log c &= 0.06213 \\ \therefore\ c &= 1.1538. \end{aligned}$$

Thus, there are two solutions. See Case III.

Ex. 2. Given $a = 31.239$, $b = 49.0012$, $A = 32° 18'$; find B, C, c.

Ans. $B = 56° 56' 56''$, or $123° 3' 4''$;
$C = 90° 45' 4''$, or $24° 38' 56''$;
$c = 58.456$, or 24.382.

74. Case III. — *Given two sides and the included angle, as a, b, C; find A, B, c.*

$$(1) \qquad \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2} \quad . \quad . \quad . \qquad (\text{Art. } 71)$$

Hence $\frac{A-B}{2}$ is known, and $\frac{A+B}{2}=90^\circ-\frac{C}{2}$.

$\therefore$ A and B are found.

$$(2) \qquad c=\frac{a\sin C}{\sin A}, \text{ or } \frac{b\sin C}{\sin B},$$

and thus c is found and the triangle solved.

In simple cases the third side c may be found directly by the formula

$$c=\sqrt{a^2+b^2-2\,ab\cos C} \quad . \quad . \quad . \qquad (\text{Art. } 56)$$

or the formula may be adapted to logarithmic calculation by the use of a subsidiary angle.

Ex. 1. Given $a=234.7$, $b=185.4$, $C=84^\circ\,36'$; find A, B, c.

Solution.

$$\begin{aligned} a &= 234.7 \\ b &= 185.4 \\ \hline a-b &= 49.3 \\ a+b &= 420.1 \\ \therefore \frac{C}{2} &= 42^\circ\,18'. \end{aligned}$$

$$\begin{aligned} \log(a-b) &= 1.69285 \\ \operatorname{colog}(a+b) &= 7.37665 \\ \log\cot\frac{C}{2} &= 10.04099 \\ \hline \log\tan\frac{A-B}{2} &= 9.11049 \\ \therefore \frac{A-B}{2} &= 7^\circ\,20'\,56''. \end{aligned}$$

$$\begin{aligned} \frac{A+B}{2} &= 47^\circ\,42'. \\ \therefore A &= 55^\circ\;\;2'\,56'', \\ B &= 40^\circ\,21'\;\;4'', \\ C &= 84^\circ\,36', \\ c &= 285.07. \end{aligned}$$

$$\begin{aligned} \log b &= 2.26811 \\ \log\sin C &= 9.99807 \\ \operatorname{colog}\sin B &= 0.18878 \\ \hline \log c &= 2.45496 \end{aligned}$$

Ex. 2. Given $a = .062387$, $b = .023475$, $C = 110° 32'$; find A, B, c.

Ans. $A = 52° 10' 33''$; $B = 17° 17' 27''$; $c = .0739635$.

75. Case IV. — *Given the three sides, as a, b, c; find A, B, C.*

The solution in this case may be performed by the formulæ of Art. 59. By means of these formulæ we may compute two of the angles, and find the third by subtracting their sum from 180°. But in practice it is better to compute the three angles independently, and check the accuracy of the work by taking their sum.

If only *one* angle is to be found, the formulæ for the *sines* or *cosines* may be used. If *all* the angles are to be found, the *tangent* formulæ are the most convenient, because then we require only the logarithms of the same four quantities, s, $s - a$, $s - b$, $s - c$, to find all the angles; whereas the sine and cosine formulæ require in addition the logs of a, b, c.

The tangent formulæ (Art. 59) may be reduced as follows:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\therefore \tan\frac{A}{2} = \frac{r}{s-a} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(Art. 62)}$$

Similarly,
$$\tan\frac{B}{2} = \frac{r}{s-b},$$

$$\tan\frac{C}{2} = \frac{r}{s-c}.$$

NOTE. — The quantity r is the radius of the inscribed circle (Art. 62).

Ex. 1. Given $a = 13$, $b = 14$, $c = 15$; find A, B, C.

Solution.

$$a = 13$$
$$b = 14$$
$$c = 15$$
$$2s = 42$$
$$s = 21.$$
$$s - a = 8,$$
$$s - b = 7,$$
$$s - c = 6.$$

$$\log(s - a) = .90309$$
$$\log(s - b) = .84510$$
$$\log(s - c) = .77815$$
$$\text{colog}\, s = 8.67778$$
$$\log r^2 = 1.20412$$
$$\log r = .60206.$$
$$\therefore \log\tan\frac{A}{2} = 9.69897.$$
$$\therefore \log\tan\frac{B}{2} = 9.75696.$$
$$\therefore \log\tan\frac{C}{2} = 9.82391.$$

$$\therefore A = 53^\circ\, 7'\, 48'';$$
$$B = 59^\circ\, 29'\, 23'';$$
$$C = 67^\circ\, 22'\, 48''.$$

Without the use of logarithms, the angles may be found by the cosine formulæ (Art. 56). These may sometimes be used with advantage, when the given lengths of a, b, c each contain less than three digits.

Ex. 2. Find the greatest angle in the triangle whose sides are 13, 14, 15.

Let $a = 15$, $b = 14$, $c = 13$. Then A is the greatest angle.

Then $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{14^2 + 13^2 - 15^2}{2 \times 14 \times 13}$$
$$= \tfrac{5}{13} = .3846 = \cos 67^\circ\, 23', \text{ nearly}$$

(by the table of natural sines).

$\therefore$ the greatest angle is 67° 23′.

EXAMPLES.

1. Given $a = 254$, $B = 16°$, $C = 64°$; find $b = 71.0919$.

2. Given $c = 338.65$, $A = 53° 24'$, $B = 66° 27'$; find $a = 313.46$.

3. Given $c = 38$, $A = 48°$, $B = 54°$; find $a = 28.87$, $b = 31.43$.

4. Given $a = 7012.5$, $B = 38° 12' 48''$, $C = 60°$; find b and c.
Ans. $b = 4382.82$; $c = 6135.94$.

5. Given $a = 528$, $b = 252$, $A = 124° 34'$; find B and C.
Ans. $B = 23° 8' 33''$; $C = 32° 17' 27''$.

6. Given $a = 170.6$, $b = 140.5$, $B = 40°$; find A and C.
Ans. $A = 51° 18' 21''$, or $128° 41' 39''$;
$C = 88° 41' 39''$, or $11° 18' 21''$.

7. Given $a = 97$, $b = 119$, $A = 50°$; find B and C.
Ans. $B = 70° \; 0' 56''$, or $109° 59' \; 4''$;
$C = 59° 59' \; 4''$, or $20° \; 0' 56''$.

8. Given $a = 7$, $b = 8$, $A = 27° 47' 45''$; find B, C, c.
Ans. $B = 32° 12' 15''$, or $147° 47' 45''$;
$C = 120°$, or $4° 24' 30''$;
$c = 13$, or 1.15385.

9. Given $b = 55$, $c = 45$, $A = 6°$; find B and C.
Ans. $B = 149° 20' 31''$; $C = 24° 39' 29''$.

10. Given $b = 131$, $c = 72$, $A = 40°$; find B and C.
Ans. $B = 108° 36' 30''$; $C = 31° 23' 30''$.

11. Given $a = 35$, $b = 21$, $C = 50°$; find A and B.
Ans. $A = 93° 11' 49''$; $B = 36° 48' 11''$.

12. Given $a = 601$, $b = 289$, $C = 100° 19' 6''$; find A and B.
Ans. $A = 56° 8' 42''$; $B = 23° 32' 12''$.

13. Given $a = 222$, $b = 318$, $c = 406$; find $A = 32° 57' 8''$.

14. Given $a = 275.35$, $b = 189.28$, $c = 301.47$; find A, B, C.
Ans. $A = 63° 30' 57''$; $B = 37° 58' 20''$; $C = 78° 30' 43''$.

15. Given $a = 5238$, $b = 5662$, $c = 9384$; find A and B.
Ans. $A = 29° 17' 16''$; $B = 31° 55' 31''$.

16. Given $a = 317$, $b = 533$, $c = 510$; find A, B, C.
Ans. $A = 35° 18' 0''$; $B = 76° 18' 52''$; $C = 68° 23' 8''$.

76. Area of a Triangle (Art. 61).

EXAMPLES.

Find the area:

1. Given $a = 116.082$, $b = 100$, $C = 118° 15' 41''$. *Ans.* 5112.3.
2. Given $a = 8$, $b = 5$, $C = 60°$. 17.320.
3. Given $b = 21.5$, $c = 30.456$, $A = 41° 22'$. 216.37.
4. Given $a = 72.3$, $A = 52° 35'$, $B = 63° 17'$. 2644.9.
5. Given $b = 100$, $A = 76° 38' 13''$, $C = 40° 5'$. 3506.8.
6. Given $a = 31.325$, $B = 13° 57' 2''$, $A = 53° 11' 18''$. *Ans.* 135.35.
7. Given $a = .582$, $b = .601$, $c = .427$. .11765.
8. Given $a = 408$, $b = 41$, $c = 401$. 8160.
9. Given $a = .9$, $b = 1.2$, $c = 1.5$. .54.
10. Given $a = 21$, $b = 20$, $c = 29$. 210.
11. Given $a = 24$, $b = 30$, $c = 18$. 216.
12. Given $a = 63.89$, $b = 138.24$, $c = 121.15$. 3869.2.

MEASUREMENT OF HEIGHTS AND DISTANCES.

77. Definitions. — One of the most important applications of Trigonometry is the determination of the heights and distances of objects which cannot be actually measured.

The actual measurement, with scientific accuracy, of a *line* of any considerable length, is a very long and difficult operation. But the accurate measurement of an *angle*, with proper instruments, can be made with comparative ease and rapidity.

By the aid of the Solution of Triangles we can determine:

(1) The distance between points which are inaccessible.

(2) The magnitude of angles which cannot be practically observed.

(3) The relative heights of distant and inaccessible points.

A *vertical line* is the line assumed by a plummet when freely suspended by a cord, and allowed to come to rest.

A *vertical plane* is any plane containing a vertical line.

A *horizontal plane* is a plane perpendicular to a vertical line.

A *vertical angle* is one lying in a vertical plane.

A *horizontal angle* is one lying in a horizontal plane.

An *angle of elevation* is a vertical angle having one side horizontal and the other ascending.

An *angle of depression* is a vertical angle having one side horizontal and the other descending.

By *distance* is meant the *horizontal* distance, unless otherwise named.

By *height* is meant the *vertical* height above or below the horizontal plane of the observer.

For a description of the requisite instruments, and the method of using them, the student is referred to books on practical surveying.*

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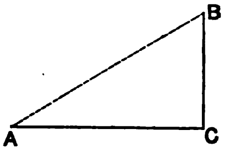
* See Johnson's Surveying, Gillespie's Surveying, Clarke's Geodesy, Gore's Geodesy, etc.

78. To find the Height of an Object standing on a Horizontal Plane, the Base of the Object being Accessible.



Let BC be a vertical object, such as a church spire or a tower.

From the base C measure a horizontal line CA.

At the point A measure the angle of elevation CAB.

We can then determine the height of the object BC; for

$$BC = AC \tan CAB.$$

EXAMPLES.

1. If $AC = 100$ feet and $CAB = 60°$, find BC.

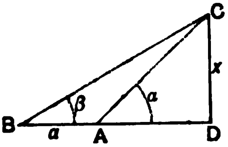
Ans. 173.2 feet.

2. If $AC = 125$ feet and $CAB = 52° 34'$, find BC.

Ans. 163.3 feet.

3. AC, the breadth of a river, is 100 feet. At the point A, on one bank, the angle of elevation of B, the top of a tree on the other bank directly opposite, is $25° 37'$; find the height of the tree. *Ans.* 47.9 feet.

79. To find the Height and Distance of an Inaccessible Object on a Horizontal Plane.



Let CD be the object, whose base D is inaccessible; and let it be required to find the height CD, and its horizontal distance from A, the nearest accessible point.

(1) At A in the horizontal line BAD observe the $\angle DAC = \alpha$; measure $AB = a$, and at B observe the $\angle DBC = \beta$.

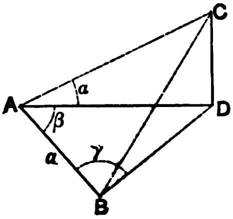
Then $$CA = \frac{a \sin \beta}{\sin (\alpha - \beta)} \quad . \ . \ . \ . \ . \ . \quad \text{(Art. 55)}$$

$$\therefore\ CD = CA \sin\alpha = \frac{a \sin\alpha \sin\beta}{\sin(\alpha-\beta)},$$

and $$AD = AC\cos\alpha = \frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}.$$

(2) *When the line BA cannot be measured directly toward the object.*

At A observe the vertical ∠ CAD $= \alpha$, and the horizontal ∠ DAB $= \beta$; measure AB $= a$, and at B observe the ∠ DBA $= \gamma$.



Then $$AD = \frac{a\sin\gamma}{\sin(\beta+\gamma)}.$$

$$\therefore\ CD = AD\tan\alpha$$

$$= \frac{a\sin\gamma\tan\alpha}{\sin(\beta+\gamma)}.$$

EXAMPLES.

1. A river 300 feet wide runs at the foot of a tower, which subtends an angle of 22° 30′ at the edge of the remote bank; find the height of the tower. *Ans.* 124.26 feet.

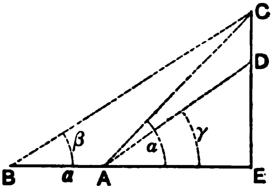
2. At 360 feet from the foot of a steeple the elevation is half what it is at 135 feet; find its height. *Ans.* 180 feet.

3. A person standing on the bank of a river observes the angle subtended by a tree on the opposite bank to be 60°, and when he retires 40 feet from the river's bank he finds the angle to be 30°; find the height of the tree and the breadth of the river. *Ans.* $20\sqrt{3}$; 20.

4. What is the height of a hill whose angle of elevation, taken at the bottom, was 46°, and 100 yards farther off, on a level with the bottom, the angle was 31°?

Ans. 143.14 yards.

80. To find the Height of an Inaccessible Object situated above a Horizontal Plane, and its Height above the Plane.



Let CD be the object, and let A and B be two points in the horizontal plane, and in the same vertical plane with CD.

At A, in the horizontal line BAE, observe the $\angle$CAE $= \alpha$, and DAE $= \gamma$; measure AB $= a$, and at B observe the $\angle$CBE $= \beta$.

Then $$\text{CE} = \frac{a \sin\alpha \sin\beta}{\sin(\alpha - \beta)} \quad \ldots\ldots\ldots \quad \text{(Art. 79)}$$

Also, $$\text{AE} = \frac{a \cos\alpha \sin\beta}{\sin(\alpha - \beta)} \quad \ldots\ldots\ldots \quad \text{(Art. 79)}$$

$$\therefore\ \text{DE} = \frac{a \cos\alpha \sin\beta \tan\gamma}{\sin(\alpha - \beta)}.$$

$$\therefore\ \text{CD} = \frac{a \sin\beta}{\sin(\alpha - \beta)}\{\sin\alpha - \cos\alpha \tan\gamma\}$$

$$= \frac{a \sin\beta \sin(\alpha - \gamma)}{\cos\gamma \sin(\alpha - \beta)}.$$

EXAMPLES.

1. A man 6 feet high stands at a distance of 4 feet 9 inches from a lamp-post, and it is observed that his shadow is 19 feet long: find the height of the lamp. *Ans.* $7\frac{1}{2}$ feet.

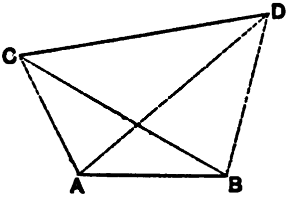
2. A flagstaff, 25 feet high, stands on the top of a cliff, and from a point on the seashore the angles of elevation of the highest and lowest points of the flagstaff are observed to be 47° 12′ and 45° 13′ respectively: find the height of the cliff. *Ans.* 348 feet.

3. A castle standing on the top of a cliff is observed from two stations at sea, which are in line with it; their

distance is a quarter of a mile: the elevation of the top of the castle, seen from the remote station, is 16° 28′; the elevations of the top and bottom, seen from the near station, are 52° 14′ and 48° 38′ respectively: (1) what is its height, and (2) what its elevation above the sea?

Ans. (1) 60.82 feet; (2) 445.23 feet.

81. To find the Distance between Two Inaccessible Objects on a Horizontal Plane.



Let C and D be the two inaccessible objects.

Measure a base line AB, from whose extremities C and D are visible. At A observe the angles CAD, DAB; and at B observe the angles CBA and CBD.

Then, in the triangle ABC, we know two angles and the side AB. ∴ AC may be found. In the triangle ABD we know two angles and the side AB. ∴ AD may be found.

Lastly, in the triangle ACD, AC and AD have been determined, and the included angle CAD has been measured; and thus CD can be found.

EXAMPLES.

1. Let AB = 1000 yards, the angles BAC, BAD = 76° 30′ and 44° 10′, respectively; and the angles ABD, ABC = 81° 12′ and 46° 5′, respectively: find the distance between C and D. *Ans.* 669.8 yards.

2. A and B are two trees on one side of a river; at two stations P and Q on the other side observations are taken, and it is found that the angles APB, BPQ, AQP are each equal to 30°, and that the angle AQB is equal to 60°. If PQ = a, show that

$$AB = \frac{a}{6}\sqrt{21}.$$

EXAMPLES.

Solve the following right triangles:

1. Given $a=51.303$, $c=150$;
find $A=20^\circ$, $B=70^\circ$, $b=140.95$.

2. Given $a=157.33$, $c=250$;
find $A=39^\circ$, $B=51^\circ$, $b=194.28$.

3. Given $a=104$, $c=185$;
find $A=34^\circ 12' 19''$, $B=55^\circ 47' 40''$, $b=153$.

4. Given $b=3$, $c=5$;
find $A=53^\circ\ 7' 48''$, $B=36^\circ 52' 11''$, $a=4$.

5. Given $b=15$, $c=17$;
find $A=28^\circ\ 4' 20''$, $B=61^\circ 55' 39''$, $a=8$.

6. Given $b=21$, $c=29$;
find $A=43^\circ 36' 10''$, $B=46^\circ 23' 49''$, $a=20$.

7. Given $c=625$, $A=44^\circ$;
find $a=434.16$, $b=449.58$.

8. Given $c=300$, $A=52^\circ$;
find $a=236.40$, $b=184.69$.

9. Given $c=13$, $A=\ 67^\circ 22' 48''$;
find $B=22^\circ 37' 11''$, $a=12$ $b=5$.

10. Given $A=77^\circ 19' 10''$, $a=40$;
find $B=12^\circ 40' 49''$, $b=9$, $c=41$.

11. Given $A=87^\circ 12' 20''$, $a=840$;
find $B=\ 2^\circ 47' 39''$, $b=41$, $c=841$.

12. Given $A=32^\circ 31' 13''$, $a=336$;
find $B=57^\circ 28' 46''$, $b=527$, $c=625$.

13. Given $A=82^\circ 41' 44''$, $a=1100$;
find $B=7^\circ 18' 16''$, $b=141$, $c=1109$.

14. Given $a=7694.5$, $b=8471$;
find $A=42^\circ 15'$, $B=47^\circ 45'$, $c=11444$.

15. Given $a=736$, $b=273$;
find $A=69^\circ 38' 56''$, $B=20^\circ 21' 3''$, $c=785$.

16. Given $a=200$, $b=609$;
find $A=18^\circ 10' 50''$, $B=71^\circ 49' 10''$, $c=641$.

17. Given $a=276$, $b=493$;
find $A=29^\circ 14' 30''$, $B=60^\circ 45' 29''$, $c=565$.

Solve the following oblique triangles:

18. Given $B=50^\circ 30'$, $C=122^\circ 9'$, $a=90$;
find $A=7^\circ 21'$, $b=542.85$, $c=595.63$.

19. Given $A=82^\circ 20'$, $B=43^\circ 20'$, $a=479$;
find $C=54^\circ 20'$, $b=331.65$, $c=392.47$.

20. Given $A=79^\circ 59'$, $B=44^\circ 41'$, $a=795$;
find $C=55^\circ 20'$, $b=567.88$, $c=663.98$.

21. Given $B=37^\circ 58'$, $C=65^\circ 2'$, $a=999$;
find $A=77^\circ 0'$, $b=630.77$, $c=929.48$.

22. Given $A=70^\circ 55'$, $C=52^\circ 9'$, $a=6412$;
find $B=56^\circ 56'$, $b=5686.0$, $c=5357.5$.

23. Given $A=48^\circ 20'$, $B=81^\circ 2' 16''$, $b=5.75$;
find $C=50^\circ 37' 44''$, $a=4.3485$, $c=4.5$.

24. Given $A=72^\circ 4'$, $B=41^\circ 56' 18''$, $c=24$;
find $C=65^\circ 59' 42''$, $a=24.995$, $b=17.559$.

25. Given $A=21^\circ 14' 25''$, $a=345$, $b=695$;
find $B=46^\circ 52' 10''$, $C=111^\circ 53' 25''$, $c=883.65$.
or $B'=133^\circ 7' 50''$, $C'=25^\circ 37' 45''$, $c'=411.92$.

26. Given $A = 41^\circ 13' \, 0''$, $a = 77.04$, $b = 91.06$;
find $B = 51^\circ \, 9' \, 6''$, $C = 87^\circ 37' 54''$, $c = 116.82$,
or $B' = 128^\circ 50' 54''$, $C' = 9^\circ 56' \, 6''$, $c' = 20.172$.

27. Given $A = 21^\circ 14' 25''$, $a = 309$, $b = 360$;
find $B = 24^\circ 57' 54''$, $C = 133^\circ 47' 41''$, $c = 615.67$,
or $B' = 155^\circ \, 2' \, 6''$, $C' = 3^\circ 43' 29''$, $c' = 55.41$.

28. Given $B = 68^\circ 10' 24''$, $a = 83.856$, $b = 85.153$;
find $A = 66^\circ \, 5' 19''$, $C = 45^\circ 44' 17''$, $c = 65.696$.

29. Given $A = 50^\circ$, $a = 119$, $b = 97$;
find $B = 38^\circ 38' 24''$, $C = 91^\circ 21' 36''$, $c = 155.3$.

30. Given $C = 65^\circ 59'$, $a = 25$, $c = 24$;
find $A = 72^\circ \, 4' 48''$, $B = 41^\circ 56' 12''$, $b = 17.56$,
or $A' = 107^\circ 55' 12''$, $B' = 6^\circ \, 5' 48''$.

31. Given $A = 18^\circ 55' 28''$, $a = 13$, $b = 37$;
find $B = 67^\circ 22' 48''$, or $B' = 112^\circ 37' 11''$.

32. Given $C = 15^\circ 11' 21''$, $a = 232$, $b = 229$;
find $A = 85^\circ 11' 58''$, $B = 79^\circ 36' 40''$, $c = 61$.

33. Given $C = 126^\circ 12' 14''$, $a = 5132$, $b = 3476$;
find $A = 32^\circ 28' 19''$, $B = 21^\circ 19' 27''$, $c = 7713.3$.

34. Given $C = 55^\circ 12' \, 3''$, $a = 20.71$, $b = 18.87$;
find $A = 67^\circ 28' 51''$, $B = 57^\circ 19' 5''$, $c = 18.41$.

35. Given $C = 12^\circ 35' \, 8''$, $a = 8.54$, $b = 6.39$;
find $A = 136^\circ 15' 48''$, $B = 31^\circ \, 9' \, 4''$, $c = 2.69$.

36. Given $C = 34^\circ \, 9' 16''$, $a = 3184$, $b = 917$;
find $A = 133^\circ 51' 34''$, $B = 11^\circ 59' 10''$, $c = 2479.2$.

37. Given $C = 32^\circ 10' 53''$, $a = 101$, $b = 29$;
find $A = 136^\circ 23' 49''$, $B = 11^\circ 25' 16''$, $c = 78$.

38. Given $C = 96^\circ 57' 20''$, $a = 401$, $b = 41$;
find $A = 77^\circ 19' 10''$, $B = 5^\circ 43' 29''$, $c = 408$.

39. Given $C=30^\circ 40' 35''$, $a=221$, $b=149$;
find $A=110^\circ 0' 57''$, $B=39^\circ 18' 27''$, $c=120$.

40. Given $a=289$, $b=601$, $c=712$;
find $A=23^\circ 32' 12''$, $B=56^\circ\ 8' 42''$, $C=100^\circ 19'\ 6''$.

41. Given $a=17$, $b=113$, $c=120$;
find $A=7^\circ 37' 42''$, $B=61^\circ 55' 38''$, $C=110^\circ 26' 40''$.

42. Given $a=15.47$, $b=17.39$, $c=22.88$;
find $A=42^\circ 30' 44''$, $B=49^\circ 25' 49''$, $C=88^\circ 3' 27''$.

43. Given $a=5134$, $b=7268$, $c=9313$;
find $A=33^\circ 15' 39''$, $B=50^\circ 56'\ 0''$, $C=95^\circ 48' 21''$.

44. Given $a=99$, $b=101$, $c=158$;
find $A=37^\circ 22' 19''$, $B=38^\circ 15' 41''$, $C=104^\circ 22' 0''$.

45. Given $a=11$, $b=13$, $c=16$;
find $A=43^\circ\ 2' 56''$, $B=53^\circ 46' 44''$, $C=83^\circ 10' 20''$.

46. Given $a=25$, $b=26$, $c=27$;
find $A=56^\circ 15'\ 4''$, $B=59^\circ 51' 10''$, $C=63^\circ 53' 46''$.

Find the area:

47. Given $a=10$, $b=12$, $C=60^\circ$. *Ans.* $30\sqrt{3}$.
48. " $a=40$, $b=60$, $C=30^\circ$. 600.
49. " $b=7$, $c=5\sqrt{2}$, $A=135^\circ$. $17\frac{1}{2}$.
50. " $a=32.5$, $b=56.3$, $C=47^\circ 5' 30''$. 670.
51. " $b=149$, $A=70^\circ 42' 30''$, $B=39^\circ 18' 28''$. 15540.
52. " $c=8.025$, $B=100^\circ\ 5' 23''$, $C=31^\circ\ 6' 12''$. 46.177.
53. " $a=5$, $b=6$, $c=7$. $6\sqrt{6}$.
54. " $a=409$, $b=169$, $c=510$. 30600.
55. " $a=577$, $b=73$, $c=520$. 12480.
56. " $a=7.152$, $b=8.263$, $c=9.375$. 28.4771.

HEIGHTS AND DISTANCES.

57. At a point 200 feet in a horizontal line from the foot of a tower, the angle of elevation of the top of the tower is observed to be 60°: find the height of the tower.

Ans. 346 feet.

58. From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff, is observed to be 30°: find the height of the cliff.

Ans. 86.6 feet.

59. From the top of a tower 117 feet high, the angle of depression of the top of a house 37 feet high is observed to be 30°: how far is the top of the house from the tower?

Ans. 138.5 feet.

60. The shadow of a tower in the sunlight is observed to be 100 feet long, and at the same time the shadow of a lamp-post 9 feet high is observed to be $3\sqrt{3}$ feet long: find the angle of elevation of the sun, and height of the tower.

Ans. 60°; 173.2 feet.

61. A flagstaff 25 feet high stands on the top of a house; from a point on the plain on which the house stands, the angles of elevation of the top and bottom of the flagstaff are observed to be 60° and 45° respectively: find the height of the house above the point of observation.

Ans. 34.15 feet.

62. From the top of a cliff 100 feet high, the angles of depression of two ships at sea are observed to be 45° and 30° respectively; if the line joining the ships points directly to the foot of the cliff, find the distance between the ships.

Ans. 73.2.

63. A tower 100 feet high stands on the top of a cliff; from a point on the sand at the foot of the cliff the angles

of elevation of the top and bottom of the tower are observed to be 75° and 60° respectively: find the height of the cliff.
Ans. 86.6 feet.

64. A man walking a straight road observes at one milestone a house in a direction making an angle of 30° with the road, and at the next milestone the angle is 60°: how far is the house from the road? *Ans.* 1524 yds.

65. A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank makes with AB an angle of 30°. He then goes 400 yards along the bank to B and finds that BC makes with BA an angle of 60°: find the breadth of the river. *Ans.* 173.2 yards.

66. From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 feet high at the foot of the hill are observed to be 45° 13′ and 47° 12′ respectively: find the height of the hill. *Ans.* 373 feet.

67. From each of two stations, east and west of each other, the altitude of a balloon is observed to be 45°, and its bearings to be respectively N.W. and N.E.; if the stations be 1 mile apart, find the height of the balloon.
Ans. 3733 feet.

68. The angle of elevation of a balloon from a station due south of it is 60°, and from another station due west of the former and distant a mile from it is 45°: find the height of the balloon. *Ans.* 6468 feet.

69. Find the height of a hill, the angle of elevation at its foot being 60°, and at a point 500 yards from the foot along a horizontal plane 30°. *Ans.* $250\sqrt{3}$ yards.

70. A tower 51 feet high has a mark at a height of 25 feet from the ground: find at what distance from the foot the two parts subtend equal angles. *Ans.* $25\sqrt{51}$ feet.

71. The angles of a triangle are as $1:2:3$, and the perpendicular from the greatest angle on the opposite side is 30 yards: find the sides. *Ans.* $20\sqrt{3}$, 60, $40\sqrt{3}$.

72. From a station B at the foot of an inclined plane BC the angle of elevation of the summit A of a mountain is 60°, the inclination of BC is 30°, the angle BCA 135°, and the length of BC is 1000 yards: find the height of A over B. *Ans.* $500(3+\sqrt{3})$ yards.

73. A right triangle rests on its hypotenuse, the length of which is 100 feet; one of the angles is 36°, and the inclination of the plane of the triangle to the horizon is 60°: find the height of the vertex above the ground. *Ans.* $25\sqrt{3}\cos 18°$.

74. The angles of depression of the top and bottom of a column observed from a tower 108 feet high are 30° and 60° respectively: find the height of the column. *Ans.* 72 feet.

75. At the foot of a mountain the elevation of its summit is found to be 45°. After ascending for one mile, at a slope of 15°, towards the summit, its elevation is found to be 60°: find the height of the mountain.

Ans. $\frac{\sqrt{3}+1}{\sqrt{2}}$ miles.

76. From the top of a cliff the angles of depression of the top and bottom of a lighthouse 97.25 feet high are observed to be 23° 17′ and 24° 19′ respectively: how much higher is the cliff than the lighthouse? *Ans.* 1942 feet.

77. A person standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51°; and when he retires 30 feet from the river's bank he observes the elevation to be 46°: find the breadth of the river. *Ans.* 155.823 feet.

78. From the top of a hill I observe two milestones on the level ground in a straight line before me, and I find

their angles of depression to be respectively 5° and 15°: find the height of the hill. *Ans.* 228.6307 yards.

79. On the bank of a river there is a column 200 feet high supporting a statue 30 feet high. The statue to an observer on the opposite bank subtends an equal angle with a man 6 feet high standing at the base of the column: find the breadth of the river. *Ans.* $10\sqrt{115}$ feet.

80. A man walking along a straight road at the rate of 3 miles an hour, sees in front of him, at an elevation of 60°, a balloon which is travelling horizontally in the same direction at the rate of 6 miles an hour; ten minutes after he observes that the elevation is 30°: prove that the height of the balloon above the road is $440\sqrt{3}$ yards.

81. The angle of elevation of a tower at a distance of 20 yards from its foot is three times as great as the angle of elevation 100 yards from the same point: show that the height of the tower is $\frac{300}{\sqrt{7}}$ feet.

82. From a window it is observed that the angle of elevation of the top of a house on the opposite side of the street is 29°, and the angle of depression of the bottom of the house is 56°: find the height of the house, supposing the breadth of the street to be 80 feet. *Ans.* 162.95 feet.

83. A and B are two positions on opposite sides of a mountain; C is a point visible from A and B; AC and BC are 10 miles and 8 miles respectively, and the angle BCA is 60°: prove that the distance between A and B is 9.165 miles.

84. P and Q are two inaccessible objects; a straight line AB, in the same plane as P and Q, is measured, and found to be 280 yards; the angle PAB is 95°, the angle QAB is $47\frac{1}{2}$°, the angle QBA is 110°, and the angle PBA is 52° 20′: find the length of PQ. *Ans.* 509.77 yards.

PART II.

SPHERICAL TRIGONOMETRY.

CHAPTER VI.

FORMULÆ RELATIVE TO SPHERICAL TRIANGLES.

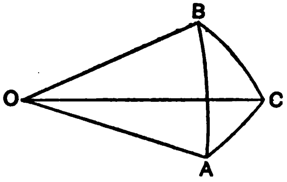
82. *Spherical Trigonometry* has for its object the solution of spherical triangles.

A *spherical triangle* is the figure formed by joining any three points on the surface of a sphere by arcs of great circles. The three points are called the *vertices* of the triangle; the three arcs are called the *sides* of the triangle.

Any two points on the surface of a sphere can be joined by two distinct arcs, which together make up a great circle passing through the points. Hence, when the points are not diametrically opposite, these arcs are unequal, one of them being less, the other greater, than 180°. It is not necessary to consider triangles in which a side is greater than 180°, since we may always replace such a side by the remaining arc of the great circle to which it belongs.

83. Geometric Principles. — It is shown in geometry (Art. 702), that if the vertex of a triedral angle is made the centre of a sphere, then the planes which form the triedral angle will cut the surface of the sphere in three arcs of great circles, forming a spherical triangle.

Thus, let O be the vertex of a triedral angle, and AOB, BOC, COA its face-angles. We may construct a sphere with its centre at O, and with any radius OA. Let AB, BC, CA be the arcs of great circles in which the planes of the face-angles AOB, BOC, COA cut the surface of this sphere; then ABC is a spherical triangle, and the arcs AB, BC, CA are its sides.



Now it is shown in geometry that the three face-angles AOB, BOC, COA are measured by the sides AB, BC, CA, respectively, of the spherical triangle, and that the diedral angles OA, OB, OC are equal to the angles A, B, C, respectively, of the spherical triangle ABC, and also that a diedral angle is measured by its plane angle.

There is then a correspondence between the triedral angle O–ABC and the spherical triangle ABC: the six parts of the triedral angle are represented by the corresponding six parts of the spherical triangle, and all the relations among the parts of the former are the same as the relations among the corresponding parts of the latter.

84. Fundamental Definitions and Properties. — The following definitions and properties are from Geometry, Book VIII.:

In every spherical triangle

Each side is less than the sum of the other two.

The sum of the three sides lies between 0° and 360°.

The sum of the three angles lies between 180° and 540°.

Each angle is greater than the difference between 180° and the sum of the other two.

If two sides are equal, the angles opposite them are equal; and conversely.

If two sides are unequal, the greater side lies opposite the greater angle; and conversely.

The perpendicular from the vertex to the base of an isosceles triangle bisects both the vertical angle and the base.

The *axis* of a circle is the diameter of the sphere perpendicular to the plane of the circle. The *poles* of a circle are the two points in which its axis meets the surface of the sphere.

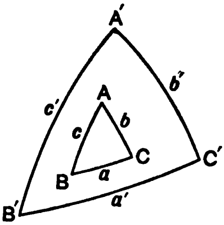
One spherical triangle is called the *polar triangle* of a second spherical triangle when the sides of the first triangle have their poles at the vertices of the second.

If the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.

Two such triangles are said to be *polar* with respect to each other. Thus:

If A'B'C' is the polar triangle of ABC, then ABC is the polar triangle of A'B'C'.

In two polar triangles, each angle of one is measured by the supplement of the corresponding side of the other. Thus:



$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c',$$
$$a = 180° - A', \quad b = 180° - B', \quad c = 180° - C'.$$

This result is of great importance; for if any general equation be established between the sides and angles of a spherical triangle, it holds of course for the polar triangle also. Hence, *by means of the above formulæ any theorem of a spherical triangle may be at once transformed into another theorem by substituting for each side and angle respectively the supplements of its opposite angle and side.*

If a spherical triangle has one right angle, it is called a *right* triangle; if it has two right angles, it is called a *bi-rectangular* triangle; and if it has three right angles, it is called a *tri-rectangular* triangle. If it has one side equal to a quadrant, it is called a *quadrantal* triangle; and if it has two sides equal to a quadrant, it is called a *bi-quadrantal* triangle.

NOTE. — It is shown in geometry that a spherical triangle may, in general, be *constructed* when any three of its six parts are given (not excepting the case in which the given parts are the three angles). In spherical trigonometry we investigate the methods by which the unknown parts of a spherical triangle may be *computed* from the above data.

EXAMPLES.

1. In the spherical triangle whose angles are A, B, C, prove

$$B+C-A<\pi \quad \dots \dots \dots \quad (1)$$
$$C+A-B<\pi \quad \dots \dots \dots \quad (2)$$
$$A+B-C<\pi \quad \dots \dots \dots \quad (3)$$

2. If C is a right angle, prove

$$A+B<\tfrac{3}{2}\pi \text{ (1), and } A-B<\frac{\pi}{2} \text{ (2).}$$

3. The angles of a triangle are A, 45°, and 120°; find the maximum value of A. *Ans.* $A<105°$.

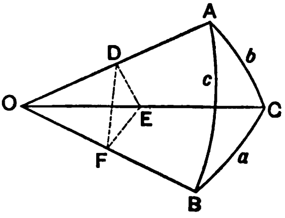
4. The angles of a triangle are A, 30°, and 150°; find the maximum value of A. *Ans.* $A<60°$.

5. The angles of a triangle are A, 20°, and 110°; find the maximum value of A. *Ans.* $A<90°$.

6. Any side of a triangle is greater than the difference between the other two.

RIGHT SPHERICAL TRIANGLES.

85. Formulæ for Right Triangles. — Let ABC be a spherical triangle in which C is a right angle, and let O be the centre of the sphere; then will OA, OB, OC be radii: let a, b, c denote the sides of the triangle opposite the angles A, B, C, respectively; then a, b, and c are the measures of the angles BOC, COA, and AOB.



From any point D in OA draw DE $\perp$ to OC, and from E draw EF $\perp$ to OB, and join DF. Then DE is $\perp$ to EF (Geom. Art. 537). Hence (Geom. Art. 507),

$$\text{DF is} \perp \text{to OB}; \quad \therefore \angle \text{DFE} = \angle \text{B} \quad . \quad . \quad \text{(Art. 83)}$$

Now $\frac{OF}{OD} = \frac{OF}{OE} \cdot \frac{OE}{OD}$; that is, $\cos c = \cos a \cos b$ . . (1)

$\frac{DE}{OD} = \frac{DE}{DF} \cdot \frac{DF}{OD}$; that is, $\sin b = \sin B \sin c$ . . (2)

Interchanging a's and b's, $\sin a = \sin A \sin c$ . . (3)

$\frac{EF}{OF} = \frac{EF}{DF} \cdot \frac{DF}{OF}$; that is, $\tan a = \cos B \tan c$ . . (4)

Interchanging a's and b's, $\tan b = \cos A \tan c$ . . (5)

$\frac{DE}{OE} = \frac{DE}{EF} \cdot \frac{EF}{OE}$; that is, $\tan b = \tan B \sin a$ . . (6)

Interchanging a's and b's, $\tan a = \tan A \sin b$ . . (7)

Multiply (6) and (7) together, and we get

$$\tan A \tan B = \frac{1}{\cos a \cos b} = \frac{1}{\cos c}, \text{ by (1).}$$

$$\therefore \cos c = \cot A \cot B \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Multiply crosswise (3) and (4), and we get

$$\sin a \cos B \tan c = \tan a \sin A \sin c.$$

$$\therefore \cos B = \frac{\sin A \cos c}{\cos a} = \sin A \cos b, \text{ by (1)} \quad . \quad . \quad . \quad (9)$$

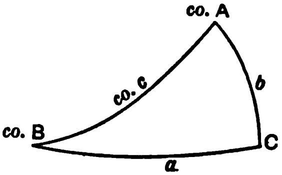
Interchanging a's and b's,

$$\cos A = \sin B \cos a \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Sch. By these ten formulæ, every case of right triangles can be solved; for every one of these ten formulæ is a distinct combination, involving three out of the five quantities, a, b, c, A, B, and there can be but ten combinations in all. Hence, any two of the five quantities being given and a third required, that third quantity may be determined by some one of the above ten formulæ.

86. Napier's Rules.—The ten preceding formulæ, which may be found difficult to remember, have been included under two simple rules, called after their inventor, *Napier's Rules of the Circular Parts.*

Let ABC be a right spherical triangle. Omit the right angle C. Then the two sides a and b, which include the right angle, the complement of the hypotenuse c, and the complements of the oblique angles A and B, are called the *circular parts* of the triangle. Thus, there are *five* circular parts, arranged in the figure in the following order: a, b, *co.* A, *co.* c, *co.* B.



Any one of these five parts may be selected and called the *middle part;* then the two parts next to it are called *adjacent parts*, and the remaining two parts are called *opposite parts.* Thus, if *co.* A is selected as the middle part, then b and *co.* c are the adjacent parts, and a and *co.* B are the opposite parts.

Then Napier's Rules are.

(1) *The sine of the middle part equals the product of the tangents of the adjacent parts.*

(2) *The sine of the middle part equals the product of the cosines of the opposite parts.*

NOTE 1.—It will assist the student in remembering these rules to notice the occurrence of the vowel *i* in *sine* and *middle*, of the vowel *a* in *tangent* and *adjacent*, and of the vowel *o* in *cosine* and *opposite.*

Napier's Rules* may be made evident by taking in detail each of the five parts as middle part, and comparing the equations thus found with the formulæ of Art. 185.

Thus, let *co. c be the middle part.* The rules give

$\sin(co.\,c) = \tan(co.\,\mathrm{A})\tan(co.\,\mathrm{B})$; $\therefore\ \cos c = \cot \mathrm{A} \cot \mathrm{B}$ . . . . (8)

$\sin(co.\,c) = \cos a \cos b$; $\therefore\ \cos c = \cos a \cos b$ . . . . (1)

co. B *the middle part.*

$\sin(co.\,\mathrm{B}) = \tan a \tan(co.\,c)$; $\therefore\ \cos \mathrm{B} = \tan a \cot c$ . . . . (4)

$\sin(co.\,\mathrm{B}) = \cos b \cos(co.\,\mathrm{A})$; $\therefore\ \cos \mathrm{B} = \cos b \sin \mathrm{A}$ . . . . (9)

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* While some find these rules to be useful aids to the memory, others question their utility.

a the middle part.

$\sin a = \tan b \tan(co. \mathrm{B})$; $\quad \therefore \sin a = \tan b \cot \mathrm{B}$ . . . . (6)

$\sin a = \cos(co. \mathrm{A}) \cos(co. c)$; $\quad \therefore \sin a = \sin \mathrm{A} \sin c$ . . . . (3)

b the middle part.

$\sin b = \tan a \tan(co. \mathrm{A})$; $\quad \therefore \sin b = \tan a \cot \mathrm{A}$ . . . . (7)

$\sin b = \cos(co. c) \cos(co. \mathrm{B})$; $\quad \therefore \sin b = \sin c \sin \mathrm{B}$ . . . . (2)

co. A the middle part.

$\sin(co\ \mathrm{A}) = \tan b \tan(co. c)$; $\quad \therefore \cos \mathrm{A} = \tan b \cot c$ . . . . (5)

$\sin(co. \mathrm{A}) = \cos a \cos(co. \mathrm{B})$; $\quad \therefore \cos \mathrm{A} = \cos a \sin \mathrm{B}$ . . . . (10)

NOTE 2.—In applying these rules it is not necessary to use the notation *co. c*, *co.* A, *co.* B, since we may write at once cos *c* for sin (*co. c*), etc.

87. The Species of the Parts.—If two parts of a spherical triangle are either both less than 90° or both greater than 90°, they are said to be of the *same species*. But if one part is less than 90° and the other part is greater than 90°, they are of *different species*.

In order to determine whether the required parts are less or greater than 90°, it will be necessary carefully to observe their algebraic signs. If the required part is determined by means of its cosine, tangent, or cotangent, the algebraic sign of the result will show whether it is less or greater than 90°. But when a required part is found in terms of its sine, it will be ambiguous, since the sines are positive in both the first and second quadrants. This ambiguity, however, may generally be removed by either of the following principles:

(1) *In a right spherical triangle, either of the sides containing the right triangle is of the same species as the opposite angle.*

(2) *The three sides of a right spherical triangle* (omitting bi-rectangular or tri-rectangular triangles) *are either all acute, or else one is acute and the other two obtuse.*

The first follows from the equation

$$\cos \mathrm{A} = \cos a \sin \mathrm{B}$$

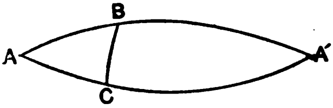
in which, since $\sin B$ is always positive ($B < 180°$), $\cos A$ and $\cos a$ must have the same sign; *i.e.*, A and a must be either both < or both > 90°.

The second follows from the equation

$$\cos c = \cos a \cos b.$$

88. Ambiguous Solution. — When the given parts of a right triangle are a side and its opposite angle, the triangle cannot be determined.

For two right spherical triangles ABC, A′BC, right angled at C, may always be found, having the angles A and A′ equal, and BC, the side opposite these angles, the same in both triangles, but the remaining sides, AB, AC, and the remaining angle ABC of the one triangle are the supplements of the remaining sides A′B, A′C, and the remaining angle A′BC of the other triangle. It is therefore ambiguous whether ABC or A′BC be the triangle required.



This ambiguity will also be found to exist, if it be attempted to determine the triangle by the equation

$$\sin b = \tan a \cot A,$$

since it cannot be determined from this equation whether the side AC is to be taken or its supplement A′C.

89. Quadrantal Triangles. — The polar triangle of a right triangle has one side a quadrant, and is therefore a *quadrantal triangle* (Art. 84). The formulæ for quadrantal triangles may be obtained by applying the ten formulæ of Art. 85 to the polar triangle. They are as follows, c being the quadrantal side:

$$\cos C = -\cos A \cos B \quad \ldots \quad (1)$$

$$\sin B = \sin b \sin C \quad \ldots \quad (2)$$

$\sin A = \sin a \sin C$ . . . . (3)

$\cos b = -\tan A \cot C$ . . . . (4)

$\cos a = -\tan B \cot C$ . . . . (5)

$\sin A = \tan B \cot b$ . . . . (6)

$\sin B = \tan A \cot a$ . . . . (7)

$\cos C = -\cot a \cot b$ . . . . (8)

$\cos b = \cos B \sin a$ . . . . (9)

$\cos a = \cos A \sin b$ . . . . (10)

EXAMPLES.

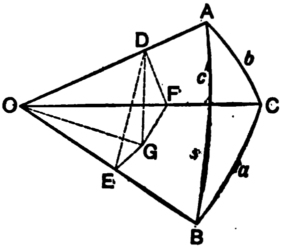
In the right triangle ABC in which the angle C is the right angle, prove the following relations:

1. $\sin^2 a + \sin^2 b - \sin^2 c = \sin^2 a \sin^2 b.$
2. $\cos^2 A \sin^2 c = \sin^2 c - \sin^2 a.$
3. $\sin^2 A \cos^2 c = \sin^2 A - \sin^2 a.$
4. $\sin^2 A \cos^2 b \sin^2 c = \sin^2 c - \sin^2 b.$
5. $2 \cos c = \cos(a + b) + \cos(a - b).$
6. $\tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a) = \tan^2 \frac{1}{2} b.$
7. $\sin^2 \frac{c}{2} = \sin^2 \frac{a}{2} \cos^2 \frac{b}{2} + \cos^2 \frac{a}{2} \sin^2 \frac{b}{2}.$
8. $\sin(c - b) = \tan^2 \frac{A}{2} \sin(c + b).$
9. If $b = c = \frac{\pi}{2}$, prove $\cos a = \cos A.$
10. If $a = b = c$, prove $\sec A = 1 + \sec a.$
11. If $c < 90°$, show that a and b are of the same species.
12. If $c > 90°$, a and b are of different species.
13. A side and the hypotenuse are of the same or opposite species, according as the included angle $<$, or $> \frac{\pi}{2}.$

OBLIQUE SPHERICAL TRIANGLES.

90. Law of Sines.—*In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*

Let ABC be a spherical triangle, O the centre of the sphere; and let a, b, c denote the sides of the triangle opposite the angles A, B, C, respectively. Then a, b, and c are the measures of the angles BOC, COA, and AOB.



From any point D in OA draw DG $\perp$ to the plane BOC, and from G draw GE, GF $\perp$ to OB, OC. Join DE, DF, and GO. Then DG is $\perp$ to GE, GF, and GO (Geom. Art. 487). Hence, DE is $\perp$ to OB, and DF $\perp$ to OC (Geom. Art. 507).

$$\therefore \angle \text{DEG} = \angle \text{B}, \text{ and } \angle \text{DFG} = \angle \text{C} \quad . \quad . \quad \text{(Art. 83)}$$

In the right plane triangles DGE, DGF, ODE, ODF,

$$\text{DG} = \text{DE} \sin \text{B} = \text{OD} \sin \text{DOE} \sin \text{B} = \text{OD} \sin c \sin \text{B},$$

$$\text{DG} = \text{DF} \sin \text{C} = \text{OD} \sin \text{DOF} \sin \text{C} = \text{OD} \sin b \sin \text{C}.$$

$$\therefore \sin c \sin \text{B} = \sin b \sin \text{C};$$

or

$$\sin b : \sin c :: \sin \text{B} : \sin \text{C}.$$

Similarly, it may be shown that

$$\sin a : \sin c :: \sin \text{A} : \sin \text{C}.$$

$$\therefore \frac{\sin a}{\sin \text{A}} = \frac{\sin b}{\sin \text{B}} = \frac{\sin c}{\sin \text{C}}.$$

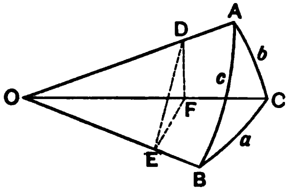
NOTE.—The common value of these three ratios is called the *modulus* of the spherical triangle.

Sch. In the figure, B, C, b, c are each less than a right angle; but it will be found on examination that the proof will hold when the figure is modified to meet any case which can occur. For example, if B alone is greater than

90°, the point G will fall outside of OB instead of between OB and OC. Then DEG will be the *supplement* of B, and thus we shall still have sin DEG = sin B.

91. Law of Cosines. — *In any spherical triangle, the cosine of each side is equal to the product of the cosines of the other two sides, plus the product of the sines of those sides into the cosine of their included angle.*

Let ABC be a spherical triangle, O the centre of the sphere, and a, b, c the sides of the triangle opposite the angles A, B, C, respectively. Then



$$a = \angle \mathrm{BOC},$$
$$b = \angle \mathrm{COA},$$
$$c = \angle \mathrm{AOB}.$$

From any point D in OA draw, in the planes AOB, AOC, respectively, the lines DE, DF ⊥ to OA. Then

$$\angle \mathrm{EDF} = \angle \mathrm{A} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Art. 83)}$$

Join EF; then in the plane triangles EOF, EDF, we have

$$\overline{\mathrm{EF}}^2 = \overline{\mathrm{OE}}^2 + \overline{\mathrm{OF}}^2 - 2\,\mathrm{OE} \cdot \mathrm{OF} \cos \mathrm{EOF} \quad . \; . \quad (1)$$

$$\overline{\mathrm{EF}}^2 = \overline{\mathrm{DE}}^2 + \overline{\mathrm{DF}}^2 - 2\,\mathrm{DE} \cdot \mathrm{DF} \cos \mathrm{EDF} \quad . \; . \quad (2)$$

also in the right triangles EOD, FOD, we have

$$\overline{\mathrm{OE}}^2 - \overline{\mathrm{DE}}^2 = \overline{\mathrm{OD}}^2, \text{ and } \overline{\mathrm{OF}}^2 - \overline{\mathrm{DF}}^2 = \overline{\mathrm{OD}}^2 \quad . \quad (3)$$

Subtracting (2) from (1), and reducing by (3), and transposing, we get

$$2\,\mathrm{OE} \cdot \mathrm{OF} \cos \mathrm{EOF} = 2\,\overline{\mathrm{OD}}^2 + 2\,\mathrm{DE} \cdot \mathrm{DF} \cos \mathrm{EDF}.$$

$$\therefore \cos \mathrm{EOF} = \frac{\mathrm{OD}}{\mathrm{OF}} \cdot \frac{\mathrm{OD}}{\mathrm{OE}} + \frac{\mathrm{DF}}{\mathrm{OF}} \cdot \frac{\mathrm{DE}}{\mathrm{OE}} \cos \mathrm{EDF},$$

or

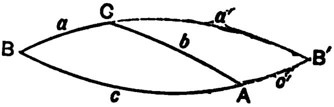
$$\cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A} \quad (4)$$

By treating the other edges in order in the same way, or by advancing letters (see Note, Art. 56) we get

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \quad . \quad . \quad (5)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad . \quad . \quad (6)$$

Sch. Formula (4) has been proved only for the case in which the sides b and c are less than quadrants; but it may be shown to be true when these sides are not less than quadrants, as follows:



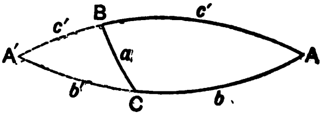
(1) Suppose c is greater than 90°. Produce BA, BC to meet in B′, and put AB′ $= c'$, CB′ $= a'$.

Then, from the triangle AB′C, we have by (4)

$$\cos a' = \cos b \cos c' + \sin b \sin c' \cos B'AC,$$

or $\cos(\pi - a) = \cos b \cos(\pi - c) + \sin b \sin(\pi - c) \cos(\pi - A).$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A.$$



(2) Suppose both b and c to be greater than 90°. Produce AB, AC to meet in A′, and put A′B $= c'$, A′C $= b'$.

Then, from the triangle A′BC, we have by (4)

$$\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

but $$b' = \pi - b, \; c' = \pi - c, \; A' = A.$$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

The triangle A′BC is called the *colunar* triangle of ABC.

92. Relation between a Side and the Three Angles. — *In any spherical triangle* ABC,

$$\cos A = - \cos B \cos C + \sin B \sin C \cos a.$$

Let A′B′C′ be the polar triangle of ABC, and denote its angles and sides by A', B', C', a', b', c'; then we have by (4) of Art. 91

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

but $$a' = \pi - A,\ b' = \pi - B,\ c' = \pi - C, \text{ etc.} \quad . \quad \text{(Art. 84)}$$

Hence, substituting, we get

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad . \quad . \quad . \quad . \quad (1)$$

Similarly,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b \quad . \quad . \quad . \quad . \quad (2)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \quad . \quad . \quad . \quad . \quad (3)$$

Rem. — **This process is called "applying the formula to the polar triangle." By means of the polar triangle, any formula of a spherical triangle may be immediately transformed into another, in which angles take the place of sides, and sides of angles.**

93. To show that in a spherical triangle ABC,

$$\cot a \sin b = \cot A \sin C + \cos C \cos b.$$

Multiply (6) of Art. 91 by $\cos b$, and substitute the result in (4) of Art. 91, and we get

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A.$$

Transpose $\cos a \cos^2 b$, and divide by $\sin a \sin b$; thus,

$$\cot a \sin b = \cos b \cos C + \frac{\sin c \cos A}{\sin a}$$

$$= \cos b \cos C + \cot A \sin C \quad . \quad \text{(by Art. 90)}$$

By interchanging the letters, we obtain five other formulæ like the preceding one. The six formulæ are as follows:

$$\cot a \sin b = \cot A \sin C + \cos C \cos b \quad . \quad . \quad . \quad . \quad (1)$$

$$\cot a \sin c = \cot A \sin B + \cos B \cos c \quad . \quad . \quad . \quad . \quad (2)$$

$$\cot b \sin a = \cot B \sin C + \cos C \cos a \quad . \quad . \quad . \quad . \quad (3)$$

$$\cot b \sin c = \cot B \sin A + \cos A \cos c \quad . \quad . \quad . \quad . \quad (4)$$

$$\cot c \sin a = \cot C \sin B + \cos B \cos a \quad . \quad . \quad . \quad . \quad (5)$$

$$\cot c \sin b = \cot C \sin A + \cos A \cos b \quad . \quad . \quad . \quad . \quad (6)$$

EXAMPLES.

1. If a, b, c be the sides of a spherical triangle, a', b', c' the sides of its polar triangle, prove

$$\sin a : \sin b : \sin c = \sin a' : \sin b' : \sin c'.$$

2. If the bisector AD of the angle A of a spherical triangle divide the side BC into the segments $\mathrm{CD} = b'$, $\mathrm{BD} = c'$, prove

$$\sin b : \sin c = \sin b' : \sin c'.$$

3. If D be any point of the side BC, prove that

$$\cot \mathrm{AB} \sin \mathrm{DAC} + \cot \mathrm{AC} \sin \mathrm{DAB} = \cot \mathrm{AD} \sin \mathrm{BAC}.$$

$$\cot \mathrm{ABC} \sin \mathrm{DC} + \cot \mathrm{ACB} \sin \mathrm{BD} = \cot \mathrm{ADB} \sin \mathrm{BC}.$$

4. If α, β, γ be the perpendiculars of a triangle, prove that

$$\sin a \sin \alpha = \sin b \sin \beta = \sin c \sin \gamma.$$

5. In Ex. 4 prove that

$$\sin a \cos \alpha = \sqrt{\cos^2 b + \cos^2 c - 2 \cos a \cos b \cos c}.$$

94. Useful Formulæ.—Several other groups of useful formulæ are easily obtained from those of Art. 91; the following are left as exercises for the student:

$$\sin a \cos \mathrm{B} = \cos b \sin c - \sin b \cos c \cos \mathrm{A} \quad . \quad . \quad (1)$$

$$\sin a \cos \mathrm{C} = \sin b \cos c - \cos b \sin c \cos \mathrm{A} \quad . \quad . \quad (2)$$

$$\sin b \cos \mathrm{A} = \cos a \sin c - \sin a \cos c \cos \mathrm{B} \quad . \quad . \quad (3)$$

$$\sin b \cos \mathrm{C} = \sin a \cos c - \cos a \sin c \cos \mathrm{B} \quad . \quad . \quad (4)$$

$$\sin c \cos \mathrm{A} = \cos a \sin b - \sin a \cos b \cos \mathrm{C} \quad . \quad . \quad (5)$$

$$\sin c \cos \mathrm{B} = \sin a \cos b - \cos a \sin b \cos \mathrm{C} \quad . \quad . \quad (6)$$

Applying these six formulæ to the polar triangle, we obtain the following six:

$$\sin A \cos b = \cos B \sin C + \sin B \cos C \cos a \quad . \quad . \quad (7)$$

$$\sin A \cos c = \sin B \cos C + \cos B \sin C \cos a \quad . \quad . \quad (8)$$

$$\sin B \cos a = \cos A \sin C + \sin A \cos C \cos b \quad . \quad . \quad (9)$$

$$\sin B \cos c = \sin A \cos C + \cos A \sin C \cos b \quad . \quad . \quad (10)$$

$$\sin C \cos a = \cos A \sin B + \sin A \cos B \cos c \quad . \quad . \quad (11)$$

$$\sin C \cos b = \sin A \cos B + \cos A \sin B \cos c \quad . \quad . \quad (12)$$

95. Formulæ for the Half Angles. — *To express the sine, cosine, and tangent of half an angle of a spherical triangle in terms of the sides.*

I. By (4) of Art. 91 we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = 1 - 2\sin^2\frac{A}{2} \quad \text{(Art. 45)}$$

$$\therefore \; 2\sin^2\frac{A}{2} = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b \sin c}.$$

$$\therefore \; \sin^2\frac{A}{2} = \frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b \sin c} \quad \text{(Art. 41)}$$

Let $2s = a + b + c$; so that s is half the sum of the sides of the triangle; then

$$a + b - c = 2(s-c), \text{ and } a - b + c = 2(s-b).$$

$$\therefore \; \sin^2\frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}.$$

$$\therefore \; \sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \quad . \quad . \quad . \quad (1)$$

Advancing letters,

$$\sin\frac{B}{2}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}} \quad . \quad . \quad . \quad (2)$$

$$\sin\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}} \quad . \quad . \quad . \quad (3)$$

II. $$2\cos^2\frac{A}{2}=1+\cos A \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 45)$$

$$=1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{\cos a-\cos(b+c)}{\sin b\sin c}.$$

$$\therefore\ \cos^2\frac{A}{2}=\frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b\sin c}$$

$$=\frac{\sin s\sin(s-a)}{\sin b\sin c}.$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Advancing letters,

$$\cos\frac{B}{2}=\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cos\frac{C}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

III. By division, we obtain

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} \quad . \quad . \quad . \quad (7)$$

$$\tan\frac{B}{2}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}} \quad . \quad . \quad . \quad (8)$$

$$\tan\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}} \quad . \quad . \quad . \quad (9)$$

Sch. The positive sign must be given to the radicals in each case in this article, because $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$ are each less than 90°.

Cor. 1.
$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{\sin(s-c)}{\sin s} \quad . \ . \ . \ . \ . \ (10)$$

$$\tan\frac{B}{2}\tan\frac{C}{2}=\frac{\sin(s-a)}{\sin s} \quad . \ . \ . \ . \ . \ (11)$$

$$\tan\frac{C}{2}\tan\frac{A}{2}=\frac{\sin(s-b)}{\sin s} \quad . \ . \ . \ . \ . \ (12)$$

Cor. 2. Since $\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$,

$$\therefore \sin A=\frac{2\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{\sin b\sin c} \quad . \ (13)$$

$$=\frac{2n}{\sin b\sin c} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (14)$$

where $n^2=\sin s\sin(s-a)\sin(s-b)\sin(s-c)$.

EXAMPLES.

1. Prove $\sin^2 A=\frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}{\sin^2 b\sin^2 c}$

$$=\frac{4n^2}{\sin^2 b\sin^2 c},$$

where $4n^2=1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c$.

2. Prove $\cos c=\cos(a+b)\sin^2\frac{C}{2}+\cos(a-b)\cos^2\frac{C}{2}$.

3. Prove $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin a\sin b\sin c}$.

4. Prove $\frac{\cos A+\cos B}{1-\cos C}=\frac{\sin(a+b)}{\sin c}$.

96. Formulæ for the Half Sides. — *To express the sine, cosine, and tangent of half a side of a spherical triangle in terms of the angles.*

By (1) of Art. 92, we have

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} = 1 - 2\sin^2\frac{a}{2} \quad . \quad \text{(Art. 45)}$$

$$\therefore\ 2\sin^2\frac{a}{2} = -\frac{\cos A + \cos(B + C)}{\sin B \sin C}.$$

$$\therefore\ \sin^2\frac{a}{2} = -\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\sin B \sin C} \quad \text{(Art. 41)}$$

Let $2S = A + B + C$; then $B + C - A = 2(S - A)$.

Proceeding in the same way as in Art. 95, we find the following expressions for the sides, in terms of the three angles:

$$\sin\frac{a}{2} = \sqrt{-\frac{\cos S \cos(S - A)}{\sin B \sin C}} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

$$\sin\frac{b}{2} = \sqrt{-\frac{\cos S \cos(S - B)}{\sin C \sin A}} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

$$\sin\frac{c}{2} = \sqrt{-\frac{\cos S \cos(S - C)}{\sin A \sin B}} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S - B)\cos(S - C)}{\sin B \sin C}} \quad . \ . \ . \ . \ . \ . \quad (4)$$

$$\cos\frac{b}{2} = \sqrt{\frac{\cos(S - C)\cos(S - A)}{\sin C \sin A}} \quad . \ . \ . \ . \ . \ . \quad (5)$$

$$\cos\frac{c}{2} = \sqrt{\frac{\cos(S - A)\cos(S - B)}{\sin A \sin B}} \quad . \ . \ . \ . \ . \ . \quad (6)$$

$$\tan\frac{a}{2} = \sqrt{-\frac{\cos S \cos(S - A)}{\cos(S - B)\cos(S - C)}} \quad . \ . \ . \ . \ . \quad (7)$$

$$\tan\frac{b}{2} = \sqrt{-\frac{\cos S \cos(S - B)}{\cos(S - C)\cos(S - A)}} \quad . \ . \ . \ . \ . \quad (8)$$

$$\tan\frac{c}{2} = \sqrt{-\frac{\cos S \cos(S - C)}{\cos(S - A)\cos(S - B)}} \quad . \ . \ . \ . \ . \quad (9)$$

Sch. 1. These formulæ may also be obtained immediately from those of Art. 95 by means of the polar triangle.

Sch. 2. The positive sign must be given to the above radicals, because $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$, are each less than 90°.

Sch. 3. These values of the sines, cosines, and tangents of the half sides are always *real.*

For S is $> 90°$ and $< 270°$ (Art. 84), so that cos S is always *negative.*

Also, in the polar triangle, any side is less than the sum of the other two (Art. 84).

$$\therefore \ \pi - A < \pi - B + \pi - C.$$

$$\therefore \ B + C - A < \pi. \qquad \therefore \ \cos(S - A) \text{ is } \textit{positive.}$$

Similarly, $\cos(S - B)$ and $\cos(S - C)$ are positive.

Cor. Since $\sin a = 2 \sin \frac{a}{2} \cos \frac{a}{2}$,

$$\therefore \ \sin a = \frac{2\sqrt{-\cos S \cos(S - A) \cos(S - B) \cos(S - C)}}{\sin B \sin C} \qquad (10)$$

$$= \frac{2N}{\sin B \sin C},$$

where $N = \sqrt{-\cos S \cos(S - A) \cos(S - B) \cos(S - C)}$.

EXAMPLES.

1. Prove $\cos C = -\cos(A + B) \cos^2 \frac{c}{2} - \cos(A - B) \sin^2 \frac{c}{2}$.

2. Prove $\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} = \frac{-N \cos S}{\sin A \sin B \sin C}$,

where $N = \sqrt{-\cos S \cos(S - A) \cos(S - B) \cos(S - C)}$.

97. Napier's Analogies.

Let
$$m = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} \quad \text{(Art. 90)} \quad (1)$$

$$= \frac{\sin A + \sin B}{\sin a + \sin b} \quad \text{(Algebra)} \quad (2)$$

or
$$= \frac{\sin A - \sin B}{\sin a - \sin b} \quad . \ . \ . \quad (3)$$

$$\cos A + \cos B \cos C = \sin B \sin C \cos a \quad \text{(Art. 92)}$$

$$= m \sin C \sin b \cos a, \text{ by (1)} \quad (4)$$

and
$$\cos B + \cos C \cos A = \sin C \sin A \cos b$$

$$= m \sin C \sin a \cos b \quad . \ . \quad (5)$$

$$\therefore (\cos A + \cos B)(1 + \cos C) = m \sin C \sin (a + b) \quad . \quad (6)$$

Dividing (2) by (6),

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin (a + b)} \cdot \frac{1 + \cos C}{\sin C}.$$

$$\therefore \tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2} \quad . \ . \quad (7)$$

(Arts. 41, 42, and 45)

Similarly,
$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2} \quad . \ . \quad (8)$$

Writing $\pi - A$ for a, etc., by Art. 84, we obtain from (7) and (8)

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{c}{2} \quad . \ . \ . \ . \quad (9)$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{c}{2} \quad . \ . \ . \ . \quad (10)$$

Sch. The formulæ (7), (8), (9), (10) are known as *Napier's Analogies,* after their discoverer. The last two may be proved without the polar triangle by starting with the formulæ of Art. 91.

Cor. In any spherical triangle whose parts are positive, and less than 180°, *the half-sum of any two sides and the half-sum of their opposite angles are of the same species.*

For, since $\cos\frac{1}{2}(a-b)$ and $\cot\frac{C}{2}$ are necessarily positive, therefore by (7) $\tan\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ are both positive or both negative.

$\therefore\ \frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are both $>$ or both $<$ or $=90°$.

98. Delambre's (or Gauss's) Analogies.

$$\sin\tfrac{1}{2}(A+B)$$

$$=\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}$$

$$=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}}$$

$$+\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}} \qquad \text{(Art. 95)}$$

$$=\frac{\sin(s-b)+\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

$$=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}}\cos\frac{C}{2} \quad \ldots\ldots \quad \text{(Arts. 41 and 95)}$$

$$\therefore\ \sin\tfrac{1}{2}(A+B)\cos\frac{c}{2}=\cos\tfrac{1}{2}(a-b)\cos\frac{C}{2} \quad \ldots \quad (1)$$

Similarly, we obtain the following three equations:

$$\sin\tfrac{1}{2}(A-B)\sin\frac{c}{2}=\sin\tfrac{1}{2}(a-b)\cos\frac{C}{2} \quad \ldots \quad (2)$$

$$\cos\tfrac{1}{2}(A+B)\cos\frac{c}{2}=\cos\tfrac{1}{2}(a+b)\sin\frac{C}{2} \quad \ldots \quad (3)$$

$$\cos\tfrac{1}{2}(A-B)\sin\frac{c}{2}=\sin\tfrac{1}{2}(a+b)\sin\frac{C}{2} \quad \ldots \quad (4)$$

Sch. 1. When the sides and angles are all less than 180°, both members of these equations are positive.

Sch. 2. Napier's Analogies may be obtained from Delambre's by division.

NOTE. — Delambre's Analogies were discovered by him in 1807, and published in the Connaissance des Temps for 1809, p. 443. They were subsequently discovered independently by Gauss, and published by him, and are sometimes improperly called Gauss's equations. Both systems may be proved geometrically. The geometric proof is the one originally given by Delambre. It was rediscovered by Professor Crofton in 1869, and published in the Proceedings of the London Mathematical Society, Vol. III. [Casey's Trigonometry, p. 41].

EXAMPLES.

In the right triangle ABC, in which C is the right angle, prove the following relations in Exs. 1–39:

1. $\sin^2 a \cos^2 b = \sin(c+b)\sin(c-b)$.

2. $\tan^2 a : \tan^2 b = \sin^2 c - \sin^2 b : \sin^2 c - \sin^2 a$.

3. $\cos^2 a \cos^2 \text{B} = \sin^2 \text{A} - \sin^2 a$.

4. $\cos^2 \text{A} + \cos^2 c = \cos^2 \text{A} \cos^2 c + \cos^2 a$.

5. $\sin^2 \text{A} - \cos^2 \text{B} = \sin^2 a \sin^2 \text{B}$.

6. If one of the sides of a right triangle be equal to the opposite angle, the remaining parts are each equal to 90°.

7. If the angle A of a right triangle be acute, show that the difference of the sides which contain it is less than 90°.

8. Prove $$\tan\frac{\text{B}}{2} = \frac{\sin(s-a)}{\sin s}.$$

9. Prove (1) $2n = \sin a \sin b$; (2) $2\text{N} = \sin a \sin \text{B}$.

10. Prove $\sin^2 a \sin^2 b = \sin^2 a + \sin^2 b - \sin^2 c$.

11. Prove $$\tan^2\frac{\text{A}}{2} = \frac{\sin(c-b)}{\sin(c+b)}.$$

12. Prove $$2\sin^2\frac{c}{2} = \sin^2\tfrac{1}{2}(a+b) + \sin^2\tfrac{1}{2}(a-b).$$

13. In a spherical triangle, if $c = 90°$, prove that

$$\tan a \tan b + \sec C = 0.$$

14. In a spherical triangle, if $c = 90°$, prove that

$$\sin^2 p = \cot \theta \cot \phi,$$

where p is the perpendicular on c, and θ and ϕ are the segments of the vertical angle.

15. Show that the ratio of the cosines of the segments of the base made by the perpendicular from the vertex is equal to the ratio of the cosines of the sides.

16. If B be the bisector of the hypotenuse, show that

$$\sin^2 B = \frac{\sin^2 a + \sin^2 b}{4 \cos^2 \frac{c}{2}}.$$

17. Prove $\tan S = \cot \frac{a}{2} \cot \frac{b}{2}.$

18. Show that $\sin \frac{A}{2} = \frac{\sqrt{\sin c + \sin a} + \sqrt{\sin c - \sin a}}{2\sqrt{\sin c}}.$

19. $\sin \frac{1}{2} A = \sqrt{\frac{\sin (c - b)}{2 \cos b \sin c}}.$

20. $\cos \frac{1}{2} A = \sqrt{\frac{\sin (c + b)}{2 \cos b \sin c}}.$

21. $\sin (a + b) \tan \frac{1}{2} (A + B) = \sin (a - b) \cot \frac{1}{2} (A - B).$

22. $\sin (A + B) = \frac{\cos a + \cos b}{1 + \cos a \cos b}.$

23. $\sin (A - B) = \frac{\cos b - \cos a}{1 - \cos a \cos b}.$

24. $\cos (A + B) = -\frac{\sin a \sin b}{1 + \cos a \cos b}.$

25. $\cos(\mathrm{A}-\mathrm{B})=\dfrac{\sin a\sin b}{1-\cos a\cos b}$.

26. $\sin^2\dfrac{c}{2}=\sin^2\dfrac{a}{2}\cos^2\dfrac{b}{2}+\cos^2\dfrac{a}{2}\sin^2\dfrac{b}{2}$.

27. $\sin(c-b)=\sin(c+b)\tan^2\dfrac{\mathrm{A}}{2}$.

28. $\sin(a-b)=\sin a\tan\dfrac{\mathrm{A}}{2}-\sin b\tan\dfrac{\mathrm{B}}{2}$.

29. $\sin(c-a)=\cos a\sin b\tan\dfrac{\mathrm{B}}{2}$.

30. If α, β be the arcs drawn from the right angle respectively perpendicular to and bisecting the hypotenuse c, show that

$$\sin^2\frac{c}{2}(1+\sin^2\alpha)=\sin^2\beta.$$

31. In a triangle, if C be a right angle and D the middle point of AB, show that

$$4\cos^2\frac{c}{2}\sin^2\mathrm{CD}=\sin^2 a+\sin^2 b.$$

In a right triangle, if p be the length of the arc drawn from the right angle C perpendicular to the hypotenuse AB, prove:

32. $\cot^2 p=\cot^2 a+\cot^2 b$.

33. $\cos^2 p=\cos^2\mathrm{A}+\cos^2\mathrm{B}$.

34. $\tan^2 a=\mp\tan a'\tan c$.

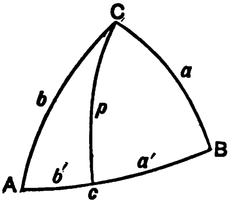
35. $\tan^2 b=\pm\tan b'\tan c$.

36. $\tan^2 a:\tan^2 b=\tan a':\tan b'$.

37. $\sin^2 p=\sin a'\sin b'$.

38. $\sin p\sin c=\sin a\sin b$.

39. $\tan a\tan b=\tan c\sin p$.



In the oblique triangle ABC, prove the following:

40. If the difference between any two angles of a triangle is 90°, the remaining angle is less than 90°.

41. If a triangle is equilateral or isosceles, its polar triangle is equilateral or isosceles.

42. If in a triangle the side $a = 90°$, show that

$$\cos A + \cos B \cos C = 0.$$

43. Prove $$4N^2 = 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C.$$

44. If p, q, r be the perpendiculars from the vertices on the opposite sides, show that

$$(1)\quad \sin a \sin p = \sin b \sin q = \sin c \sin r = 2n.$$
$$(2)\quad \sin A \sin p = \sin B \sin q = \sin C \sin r = 2N.$$

45. Prove $$\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin^2 a + \sin^2 b + \sin^2 c} = \frac{1 + \cos A \cos B \cos C}{1 - \cos a \cos b \cos c}.$$

46. If l be the length of the arc joining the middle point of the base to the vertex, find an expression for its length in terms of the sides. *Ans.* $\cos l = \dfrac{\cos a + \cos b}{2 \cos \frac{c}{2}}$.

47. If the side c of a triangle be 90°, and δ the arc drawn at right angles to it from the opposite vertex, show that

$$\cot^2 \delta = \cot^2 A + \cot^2 B.$$

48. If $A = B + C$, and D be the middle point of a, show that $a = 2\,AD$.

49. If D be the middle point of c, show that

$$\cos a + \cos b = 2 \cos \frac{c}{2} \cos CD.$$

50. If $b + c = \pi$, show that $\sin 2B + \sin 2C = 0$.

CHAPTER VII.

SOLUTION OF SPHERICAL TRIANGLES.

99. Preliminary Observations.—In every spherical triangle there are *six elements*, the three sides and the three angles, besides the radius of the sphere, which is supposed constant. *The solution of spherical triangles* is the process by which, when the values of *any three* elements are given, we calculate the values of the remaining three (Art. 84, Note).

In making the calculations, attention must be paid to the algebraic signs of the functions. When angles greater than 90° occur in calculation, we replace them by their supplements; and if the functions of such angles be either *cosine*, *tangent*, *cotangent*, or *secant*, we take account of the change of sign.

It is necessary to avoid the calculation of very small angles by their cosines, or of angles near 90° by their sines, for their tabular differences vary too slowly. It is better to determine such angles, for example, by means of their tangents.

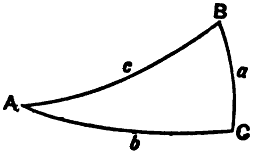
We shall begin with the right triangle; here two elements, in addition to the right angle, will be supposed known.

SOLUTION OF RIGHT SPHERICAL TRIANGLES.

100. The Solution of Right Spherical Triangles presents Six Cases, which may be solved by the formulæ of Art. 85. If the formula required for any case be not remembered, it is always easy to find it by Napier's Rules (Art.

86). In applying these rules, we must choose the middle part as follows:

When the three parts considered are all adjacent, the one between is, of course, the middle part. When only two are adjacent, the other one is the middle part.



Let ABC be a spherical triangle, right-angled at C, and let a, b, c denote the sides opposite the angles A, B, C, respectively.

We shall assume that the parts are all positive and less than 180° (Art. 82).

101. Case I. — Given the hypotenuse c and an angle A; to find a, b, B.

By (3), (5), and (8) of Art. 85, or by Napier's Rules, we have

$$\sin a = \sin c \sin A,$$

$$\tan b = \tan c \cos A,$$

$$\cot B = \cos c \tan A.$$

Since a is found by its sine, it would be ambiguous, but the ambiguity is removed because a and A are of the same species [Art. 87, (1)]. B and b are determined immediately without ambiguity.

If a be very near 90°, we commence by calculating the values of b and B, and then determine a by either of the formulæ

$$\tan a = \sin b \tan A, \quad \tan a = \tan c \cos B.$$

Check. — As a final step, in order to guard against numerical errors, it is often expedient to check the logarithmic work, which may be done in every case without the necessity of new logarithms. To check the work, we make up a formula between the three required parts, and see whether it is satisfied by the results. In the present case, when the three parts a, b, B have been found, the *check formula* is

$$\sin a = \tan b \cot B \quad . \quad . \quad . \quad . \quad [(6) \text{ of Art. } 85]$$

Ex. 1. Given $c = 81° 29' 32''$, $A = 32° 18' 17''$; find a, b, B.

Solution.

$$\begin{array}{rl}
\log \sin c & = 9.99520 \\
\log \sin A & = 9.72788 \\
\hline
\log \sin a & = 9.72308 \\
\therefore\ a & = 31° 54' 25''. \\
\log \cos c & = 9.17009 \\
\log \tan A & = 9.80092 \\
\hline
\log \cot B & = 8.97101 \\
\therefore\ B & = 84° 39' 21''.
\end{array}$$

$$\begin{array}{rl}
\log \tan c & = 10.82510 \\
\log \cos A & = 9.92697 \\
\hline
\log \tan b & = 10.75207 \\
\therefore\ b & = 79° 57' 48''.
\end{array}$$

Check.

$$\begin{array}{rl}
\log \tan b & = 10.75207 \\
\log \cot B & = 8.97101 \\
\hline
\log \sin a & = 9.72308
\end{array}$$

Ex. 2. Given $c = 110° 46' 20''$, $A = 80° 10' 30''$; find a, b, B.

Ans. $a = 67° 5' 52''$, $b = 155° 46' 42''$, $B = 153° 58' 24''$.

102. Case II. — Given the hypotenuse c **and a side** a; **to find** b, A, B.

By (1), (3), (4) of Art. 85, or by Napier's Rules, we have

$$\cos b = \frac{\cos c}{\cos a}, \ \sin A = \frac{\sin a}{\sin c}, \ \cos B = \frac{\tan a}{\tan c}.$$

The *check formula* involves b, A, B; therefore, from (9) of Art. 85 we have

$$\cos B = \sin A \cos b.$$

In this case there is an apparent ambiguity in the value of A, but this is removed by considering that A and a are always of the same species (Art. 87).

Ex. 1. Given $c = 140°$, $a = 20°$; find b, A, B.

Solution.

$$\begin{aligned}
\log \cos c &= 9.88425- \\
\operatorname{colog} \cos a &= \underline{0.02702} \\
\log \cos b &= 9.91127- \\
\therefore\ b &= 144° \, 36' \, 28''. \\
\log \tan a &= 9.56106 \\
\operatorname{colog} \tan c &= \underline{0.07619-} \\
\log \cos \mathrm{B} &= 9.63725- \\
\therefore\ \mathrm{B} &= 115° \, 42' \, 23''.
\end{aligned}$$

$$\begin{aligned}
\log \sin a &= 9.53405 \\
\operatorname{colog} \sin c &= \underline{0.19193} \\
\log \sin \mathrm{A} &= 9.72598 \\
\therefore\ \mathrm{A} &= 32° \, 8' \, 48''.
\end{aligned}$$

Check.

$$\begin{aligned}
\log \sin \mathrm{A} &= 9.72598 \\
\log \cos b &= \underline{9.91127} \\
\log \cos \mathrm{B} &= 9.63725
\end{aligned}$$

Ex. 2. Given $c = 72° \, 30'$, $a = 45° \, 15'$: find b, A, B.

Ans. $b = 64° \, 42' \, 52''$, $\mathrm{A} = 48° \, 7' \, 44''$, $\mathrm{B} = 71° \, 27' \, 15''$.

103. Case III.—Given a side a and the adjacent angle B; to find A, b, c.

By (10), (6), (4) of Art. 85, we have

$$\cos \mathrm{A} = \cos a \sin \mathrm{B}, \quad \tan b = \sin a \tan \mathrm{B}, \quad \cot c = \cot a \cos \mathrm{B}.$$

Check formula, $\cos \mathrm{A} = \tan b \cot c$.

In this case there is evidently no ambiguity.

Ex. 1. Given $a = 31° \, 20' \, 45''$, $\mathrm{B} = 55° \, 30' \, 30''$; find A, b, c.

Solution.

$$\begin{aligned}
\log \cos a &= 9.93148 \\
\log \sin \mathrm{B} &= \underline{9.91604} \\
\log \cos \mathrm{A} &= 9.84752 \\
\therefore\ \mathrm{A} &= 45° \, 15' \, 28''. \\
\log \cot a &= 0.21531 \\
\log \cos \mathrm{B} &= \underline{9.75304} \\
\log \cot c &= 9.96835 \\
\therefore\ c &= 47° \, 5' \, 11''.
\end{aligned}$$

$$\begin{aligned}
\log \sin a &= 9.71617 \\
\log \tan \mathrm{B} &= \underline{0.16300} \\
\log \tan b &= 9.87917 \\
\therefore\ b &= 37° \, 7' \, 50''.
\end{aligned}$$

Check.

$$\begin{aligned}
\log \tan b &= 9.87917 \\
\log \cot c &= \underline{9.96835} \\
\log \cos \mathrm{A} &= 9.84752
\end{aligned}$$

Ex. 2. Given $a = 112° 0' 0''$, $B = 152° 23' 1''$; find A, b, c.

Ans. $A = 100°$, $b = 154° 7' 26''$, $c = 70° 18' 10''$.

104. Case IV. — Given a side a and the opposite angle A; to find b, c, B.

By (7), (3), (10) of Art. 85, we have

$$\sin b = \tan a \cot A,\quad \sin c = \frac{\sin a}{\sin A},\quad \sin B = \frac{\cos A}{\cos a}.$$

Check formula, $\sin b = \sin c \sin B$.

In this case there is an ambiguity, as the parts are determined by their sines, and two values for each are in general admissible. But for each value of b there will, *in general*, be only one value for c, since c and b are connected by the relation $\cos c = \cos a \cos b$ (Art. 85); and at the same time only one admissible value for B, since $\cos c = \cot A \cot B$. Hence there will be, *in general*, only *two* triangles having the given parts, except when the side a is a quadrant and the angle A is also 90°, in which case the solution becomes *indeterminate*.

It is also easily seen from a figure that the ambiguity must occur in general (Art. 88).

When $a = A$, the formulæ, and also the figure, show that b, c, and B are each 90°.

Ex. 1. Given $a = 46° 45'$, $A = 59° 12'$: find b, c, B.

Solution.

$\log \tan a = 0.02655$	$\log \sin a = 9.86235$
$\log \cot A = 9.77533$	$\log \sin A = 9.93397$
$\log \sin b = 9.80188$	$\log \sin c = 9.92838$
$\therefore\ b = 39° 19' 24''$.	$\therefore\ c = 57° 59' 29''$,
or $140° 40' 36''$.	or $122° 0' 31''$.

$$\begin{array}{rl} \log\cos A &= 9.70931 \\ \log\cos a &= \underline{9.83581} \\ \log\sin B &= 9.87350 \\ \therefore\ B &= 48^\circ 21' 28'', \\ &\text{or } 131^\circ 38' 32''. \end{array}$$

Check.

$$\begin{array}{rl} \log\sin c &= 9.92838 \\ \log\sin B &= \underline{9.87350} \\ \log\sin b &= 9.80188 \end{array}$$

Ex. 2. Given $a = 112^\circ$, $A = 100^\circ$: find b, c, B.

Ans. $b = 154^\circ\ 7' 26''$, $c = 70^\circ 18' 10''$, $B = 152^\circ 23'\ 1''$.
or $25^\circ 52' 34''$, or $109^\circ 41' 50''$, or $27^\circ 36' 59''$.

105. Case V. — Given the two sides a and b; to find A, B, c.

By (7), (6), (1) of Art. 85, we have

$$\cot A = \cot a \sin b,\quad \cot B = \cot b \sin a,\quad \cos c = \cos a \cos b.$$

Check formula, $\cos c = \cot A \cot B$.

In this case there is no ambiguity.

Ex. 1. Given $a = 54^\circ 16'$, $b = 33^\circ 12'$: find A, B, c.

Ans. $A = 68^\circ 29' 53''$, $B = 38^\circ 52' 26''$, $c = 60^\circ 44' 46''$.

Ex. 2. Given $a = 56^\circ 34'$, $b = 27^\circ 18'$: find A, B, c.

Ans. $A = 73^\circ 9' 13''$, $B = 31^\circ 44' 9''$, $c = 60^\circ 41' 9''$.

106. Case VI. — Given the two angles A and B; to find a, b, and c.

By (10), (9), (8)

$$\cos a = \frac{\cos A}{\sin B},\quad \cos b = \frac{\cos B}{\sin A},\quad \cos c = \cot A \cot B.$$

Check formula, $\cos c = \cos a \cos b$.

There is no ambiguity in this case.

Ex. 1. Given $A = 74^\circ 15'$, $B = 32^\circ 10'$: find a, b, c.

Ans. $a = 59^\circ 20' 44''$, $b = 28^\circ 24' 54''$, $c = 63^\circ 21' 24''.5$.

Ex. 2. Given $A = 91^\circ 11'$, $B = 111^\circ 11'$: find a, b, c.

Ans. $a = 91^\circ 16' 8''$, $b = 111^\circ 11' 16''$, $c = 89^\circ 32' 29''$.

107. Quadrantal and Isosceles Triangles. — Since the polar triangle of a quadrantal triangle is a right triangle (Art. 84), we have only to solve the polar triangle by the formulæ of Art. 85, and take the supplements of the parts thus found for the required parts of the given triangle; or we can solve the quadrantal triangle immediately by the formulæ of Art. 89.*

A biquadrantal triangle is indeterminate unless either the base or the vertical angle be given.

An *isosceles triangle* is easily solved by dividing it into two equal right triangles by drawing an arc from the vertex to the middle of the base.

The solution of triangles in which $a + b = \pi$, or $A + B = \pi$, can be made to depend on the solution of right triangles. Thus (see the second figure of Art. 91) the triangle B′AC has the two equal sides, a' and b, given, or the two equal angles, A and B′, given, according as $a + b = \pi$ or $A + B = \pi$ in the triangle ABC.

EXAMPLES.

Solve the following right triangles:

1.	Given $c=32^\circ 34'$,	$a=44^\circ 44'$;	
	find $a=22^\circ 15' 43''$,	$b=24^\circ 24' 19''$,	$B=50^\circ 8' 21''$.
2.	Given $c=69^\circ 25' 11''$,	$A=54^\circ 54' 42''$;	
	find $a=50^\circ\ 0'\ 0''$,	$b=56^\circ 50' 49''$,	$B=63^\circ 25' 4''$.
3.	Given $c=55^\circ\ 9' 32''$,	$a=22^\circ 15'\ 7''$;	
	find $b=51^\circ 53'$,	$A=27^\circ 28' 37'.5$,	$B=73^\circ 27' 11''.1$.
4.	Given $c=127^\circ 12'$,	$a=141^\circ 11'$;	
	find $b=39^\circ\ 6' 25''$,	$A=128^\circ 5' 54''$,	$B=52^\circ 21' 49''$.
5.	Given $a=118^\circ 54'$,	$B=12^\circ 19'$;	
	find $A=95^\circ 55'\ 2''$,	$b=10^\circ 49' 17''$,	$c=118^\circ 20' 20''$.

---

* Quadrantal triangles are generally avoided in practice, but when unavoidable, they are readily solved by either of these methods.

6.	Given $a=29^\circ 46'\ 8''$,	$B=137^\circ 24'21''$;	
	find $A=54^\circ\ 1'16''$,	$b=155^\circ 27'54''$,	$c=142^\circ\ 9'13''$.
7.	Given $a=77^\circ 21'50''$,	$A=83^\circ 56'40''$;	
	find $b=28^\circ 14'31''$,	$c=78^\circ 53'20''$,	$B=\ 28^\circ 49'57''$,
	or $b'=151^\circ 45'29''$,	$c'=101^\circ 6'40''$,	$B'=151^\circ 10'\ 3''$.
8.	Given $a=68^\circ$,	$A=80^\circ$;	
	find $b=25^\circ 52'33''$,	$c=70^\circ 18'10''$,	$B=\ 27^\circ 36'59''$,
	or $b'=154^\circ 7'27''$,	$c'=109^\circ 41'50''$,	$B'=152^\circ 23'\ 1''$.
9.	Given $a=144^\circ 27'3''$,	$b=32^\circ 8'56''$;	
	find $A=126^\circ 40'24''$,	$B=47^\circ 13'43''$,	$c=133^\circ 32'26''$.
10.	Given $a=36^\circ 27'$,	$b=43^\circ 32'31''$;	
	find $A=46^\circ 59'43''$,	$B=57^\circ 59'19''$,	$c=\ 54^\circ 20'$.
11.	Given $A=63^\circ 15'12''$,	$B=135^\circ 33'39''$;	
	find $a=49^\circ 59'56''$,	$b=143^\circ 5'12''$,	$c=120^\circ 55'34''$.
12.	Given $A=67^\circ 54'47''$,	$B=99^\circ 57'35''$;	
	find $a=67^\circ 33'27''$,	$b=100^\circ 45'$,	$c=\ 94^\circ 5'$.

13. Solve the quadrantal triangle in which

$$c=90^\circ,\ A=42^\circ 1',\ B=121^\circ 20'.$$

Ans. $C=67^\circ 16'22''$, $b=112^\circ 10'20''$, $a=46^\circ 31'36''$.

14. Solve the quadrantal triangle in which

$$a=174^\circ 12'49'',\ b=94^\circ 8'20'',\ c=90^\circ.$$

Ans. $A=175^\circ 57'10''$, $B=135^\circ 42'55''$, $C=135^\circ 34'8''$.

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES.

108. The Solution of Oblique Spherical Triangles presents Six Cases; as follows:

I. *Given two sides and the included angle, a, b,* C.

II. *Given two angles and the included side,* A, B, *c.*

III. *Given two sides and an angle opposite one of them, a, b,* A.

IV. *Given two angles and a side opposite one of them,* A, B, a.

V. *Given the three sides,* a, b, c.

VI. *Given the three angles,* A, B, C.

These six cases are immediately resolved into *three pairs* of cases by the aid of the polar triangle (Art. 84).

For when two sides and the included angle are given, and the remaining parts are required, the application of the data to the polar triangle transforms the problem into the supplemental problem: given two angles and the included side, to find the remaining parts.

Similarly, cases III. and IV. are supplemental, also V. and VI.

The parts are all positive and less than 180° (Art. 82). The attention of the student is called to Art. 99.

109. Case I. — Given two sides, a, b, **and the included angle** C; **to find** A, B, c.

By Napier's Analogies, (7) and (8) of Art. 97,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{C}{2}.$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{C}{2}.$$

These determine $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, and therefore A and B; then c can be found by Art. 90, or by one of Gauss's equations (Art. 98). Since c is found from its *sine* in Art. 90, it may be uncertain which of two values is to be given to it: if we determine c from one of Gauss's equations, it is free from ambiguity. We may therefore find c from (3) of Art. 98. Thus

$$\cos \tfrac{1}{2}(A+B) \cos \frac{c}{2} = \cos \tfrac{1}{2}(a+b) \sin \frac{C}{2}.$$

Check, $\tan\frac{1}{2}(a+b)\cos\frac{1}{2}(A+B)=\cos\frac{1}{2}(A-B)\tan\frac{c}{2}$.

There is no ambiguity in this case.

Ex. 1. Given $a=43^\circ\,18'$, $b=19^\circ\,24'$, $C=74^\circ\,22'$: find A, B, c.

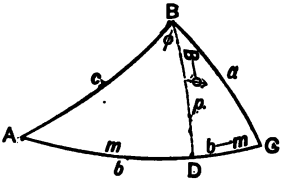
Solution.

$$\tfrac{1}{2}(a+b)=31^\circ\,21',\ \tfrac{1}{2}(a-b)=11^\circ\,57',\ \tfrac{1}{2}C=37^\circ\,11'.$$

$$\begin{aligned}
\log\cos\tfrac{1}{2}(a-b) &= 9.99048\\
\log\sec\tfrac{1}{2}(a+b) &= 0.06854\\
\log\cot\frac{C}{2} &= \underline{10.12000}\\
\log\tan\tfrac{1}{2}(A+B) &= 10.17902\\
\therefore\ \tfrac{1}{2}(A+B) &= 56^\circ\,29'\,17''\\
\tfrac{1}{2}(A-B) &= \underline{27^\circ\,41'\ \ 0''}\\
\therefore\ A &= 84^\circ\,10'\,17''\\
B &= 28^\circ\,48'\,17''.\\
c &= 41^\circ\,36'\ \ 0''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(a-b) &= 9.31609\\
\log\operatorname{cosec}\tfrac{1}{2}(a+b) &= 0.28377\\
\log\cot\frac{C}{2} &= \underline{10.12000}\\
\log\tan\tfrac{1}{2}(A-B) &= 9.71986\\
\therefore\ \tfrac{1}{2}(A-B) &= 27^\circ\,41'\,0''.\\
\log\cos\tfrac{1}{2}(a+b) &= 9.93146\\
\log\sec\tfrac{1}{2}(A+B) &= 0.25797\\
\log\sin\frac{C}{2} &= \underline{9.78130}\\
\log\cos\frac{c}{2} &= 9.97073\\
\therefore\ \frac{c}{2} &= 20^\circ\,48'\,0''.
\end{aligned}$$

Otherwise thus: Let fall the perpendicular BD, dividing the triangle ABC into two right triangles, BDA, BDC. Denote AD by m, the angle ABD by ϕ, and BD by p. Then by Napier's Rules, we have



$$\cos C=\tan(b-m)\cot a;$$

$$\sin(b-m)=\cot C\tan p;\quad \sin m=\cot A\tan p.$$

$$\therefore\ \tan(b-m)=\tan a\cos C\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (1)$$

and $$\tan A\sin m=\tan C\sin(b-m)\ .\ .\ .\ .\ .\ \ (2)$$

From (1) m is determined, and from (2) A is determined. In a similar manner B may be found.

Also, from the same triangles, we have by Napier's Rules

$$\cos a = \cos (b - m) \cos p; \quad \cos c = \cos m \cos p.$$

$$\therefore \cos c \cos (b - m) = \cos m \cos a,$$

from which c is found.

NOTE.—This method has the advantage that, in using it, nothing need be remembered except Napier's Rules.

If only the side c is wanted, it may be found from (4) of Art. 91, without previously determining A and B. This formula may be adapted to logarithms by the use of a *subsidiary angle.*

Ex. 2. Given $b = 120° 30' 30''$, $c = 70° 20' 20''$, $A = 50° 10' 10''$; find B, C, a.

Ans. $B = 135° 5' 28''$, $C = 50° 30' 8''$, $a = 69° 34' 56''$.

110. Case II.—Given two angles, A, B, **and the included side** c; **to find** a, b, C.

By Napier's Analogies (9) and (10) of Art. 97,

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{c}{2},$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{c}{2},$$

from which a and b are found.

The remaining part C may be found by (2) of Art. 98.

$$\sin \tfrac{1}{2}(a - b) \cos \frac{C}{2} = \sin \tfrac{1}{2}(A - B) \sin \frac{c}{2}.$$

Check, $\cos \tfrac{1}{2}(a - b) \cot \frac{C}{2} = \cos \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(A + B).$

There is no ambiguity in this case.

Ex. 1. Given $A = 68°40'$, $B = 56°20'$, $c = 84°30'$; find a, b, C.

Solution.

$$\tfrac{1}{2}(A+B) = 62°\,30', \ \tfrac{1}{2}(A-B) = 6°\,10', \ \frac{c}{2} = 42°\,15'.$$

$$\begin{aligned}
\log\cos\tfrac{1}{2}(A-B) &= 9.99748\\
\log\sec\tfrac{1}{2}(A+B) &= 0.33559\\
\log\tan\frac{c}{2} &= \underline{9.95825}\\
\log\tan\tfrac{1}{2}(a+b) &= 10.29132\\
\therefore\ \tfrac{1}{2}(a+b) &= 62°\,55'\ \ 9''\\
\tfrac{1}{2}(a-b) &= \underline{6°\,16'\,39''}\\
a &= 69°\,11'\,48''\\
b &= 56°\,38'\,30''.\\
C &= 97°\,19'\ \ 4''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(A-B) &= 9.03109\\
\log\operatorname{cosec}\tfrac{1}{2}(A+B) &= 0.05207\\
\log\tan\frac{c}{2} &= \underline{9.95825}\\
\log\tan\tfrac{1}{2}(a-b) &= 9.04141\\
\therefore\ \tfrac{1}{2}(a-b) &= 6°\,16'\,39''.\\
\log\sin\tfrac{1}{2}(A-B) &= 9.03109\\
\log\operatorname{cosec}\tfrac{1}{2}(a-b) &= 0.96120\\
\log\sin\frac{c}{2} &= \underline{9.82761}\\
\log\cos\frac{C}{2} &= 9.81990\\
\therefore\ \frac{C}{2} &= 48°\,39'\,32''.
\end{aligned}$$

Otherwise thus: Let fall the perpendicular BD (see last figure). Denote, as before, AD by m, the angle ABD by ϕ, and BD by p. Then by Napier's Rules, we have

$$\cos c = \cot\phi\cot A;$$

$$\cos\phi = \cot c\tan p; \ \cos(B-\phi) = \cot a\tan p.$$

$$\therefore\ \cot\phi = \tan A\cos c \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\tan a\cos(B-\phi) = \cos\phi\tan c \quad \ldots\ldots \quad (2)$$

From (1) ϕ is determined, and from (2) a is found. Similarly b may be found.

Also, from the same triangles, we have

$$\cos C\sin\phi = \cos A\sin(B-\phi),$$

from which C is found.

Ex. 2. Given

$$A = 135^\circ\, 5'\, 28'',\ C = 50^\circ\, 30'\, 8'',\ b = 69^\circ\, 34'\, 56'';$$

find a, c, B.

Ans. $a = 120^\circ\, 30'\, 30''$, $c = 70^\circ\, 20'\, 20''$, $B = 50^\circ\, 10'\, 10''$.

111. Case III. — Given two sides, a, b, and the angle A opposite one of them; to find B, C, c.

The angle B is found from the formula,

$$\sin B = \frac{\sin b}{\sin a} \sin A \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 90) \quad (1)$$

Then C and c are found from Napier's Analogies,

$$\tan\frac{C}{2} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} \cot\tfrac{1}{2}(A-B) \quad . \quad . \quad . \quad . \quad (2)$$

$$\tan\frac{c}{2} = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} \tan\tfrac{1}{2}(a-b) \quad . \quad . \quad . \quad . \quad (3)$$

Check, $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Since B is found from its sine in (1), it will have two values, if $\sin A \sin b < \sin a$, and the triangle, in general, will admit of two solutions.

When $\sin A \sin b > \sin a$, there will be no solution, for then $\sin B > 1$.

In order that either of these values found for B may be admissible, it is necessary and sufficient that, when substituted in (2) and (3), they give positive values for $\tan\frac{C}{2}$ and $\tan\frac{c}{2}$, or which is the same thing, that $A - B$ and $a - b$ have the same sign. Hence we have the following

Rule. — *If both values of* B *obtained from* (1) *be such as that* $A - B$ *and* $a - b$ *have like signs, there are two complete solutions. If only one of the values of* B *satisfies this condition, there is only one triangle that satisfies the problem, since*

in this case* C, *or* $c > 180°$. *If neither of the values of* B *makes* $A - B$ *and* $a - b$ *of the same signs, the problem is impossible.

This case is known as the *ambiguous case,* and is like the analogous ambiguity in Plane Trigonometry (Art. 73), though it is somewhat more complex. For a complete discussion of the *Ambiguous Case,* the student is referred to Todhunter's Spherical Trigonometry, pp. 53–58; McCollend and Preston's Spherical Trigonometry, pp. 137–143; Serret's Trigonometry, pp. 191–195, etc.

Ex. 1. Given $a = 42° 45'$, $b = 47° 15'$, $A = 56° 30'$: find B, C, c.

Solution.

$\log \sin b = 9.86589$	$\frac{1}{2}(a+b) = 45° \; 0' \; 0''.$
$\text{colog} \sin a = 0.16826$	$\frac{1}{2}(a-b) = -\;2° \; 15' \; 0''.$
$\log \sin A = \underline{9.92110}$	$\frac{1}{2}(A+B) = 60° \; 28' \; 2''.$
$\log \sin B = 9.95525$	$\frac{1}{2}(A-B) = -\;3° \; 58' \; 2''.$
$\therefore \; B = 64° \; 26' \; 4'',$	$\frac{1}{2}(A+B') = 86° \; 1' \; 58''.$
$B' = 115° \; 33' \; 56''.$	$\frac{1}{2}(A-B') = -\;29° \; 31' \; 58''.$

Since both values of B are such that $A - B$, $A - B'$, and $a - b$, are all negative, there are *two* solutions, by the above Rule.

(1) *When* $B = 64° \; 26' \; 4''$.

$\log \sin \frac{1}{2}(a-b) = 8.59395-$	$\log \sin \frac{1}{2}(A+B) = 9.93956$
$\text{colog} \sin \frac{1}{2}(a+b) = 0.15052$	$\text{colog} \sin \frac{1}{2}(A-B) = 1.15998-$
$\log \cot \frac{1}{2}(A-B) = \underline{1.15894-}$	$\log \tan \frac{1}{2}(a-b) = \underline{8.59428-}$
$\log \tan \frac{C}{2} = 9.90341$	$\log \tan \frac{c}{2} = 9.69382$
$\therefore \; \frac{C}{2} = 38° \; 40' \; 48''.$	$\therefore \; \frac{c}{2} = 26° \; 17' \; 40''.$
$\therefore \; C = 77° \; 21' \; 36''.$	$\therefore \; c = 53° \; 35' \; 20''.$

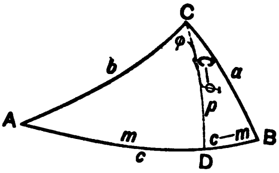
(2) *When* $B' = 115° 33' 56''$.

$$\log \sin \tfrac{1}{2}(a-b) = 8.59395-$$
$$\operatorname{colog} \sin \tfrac{1}{2}(a+b) = 0.15051$$
$$\log \cot \tfrac{1}{2}(A-B') = \underline{0.24678-}$$
$$\log \tan \frac{C'}{2} = 8.99124$$
$$\therefore \frac{C'}{2} = 5° 35' 50''.$$
$$\therefore C' = 11° 11' 40''.$$

$$\log \sin \tfrac{1}{2}(A+B') = 9.99896$$
$$\operatorname{colog} \sin \tfrac{1}{2}(A-B') = 0.30722-$$
$$\log \tan \tfrac{1}{2}(a-b) = \underline{8.59428-}$$
$$\log \tan \frac{c'}{2} = 8.90046$$
$$\therefore \frac{c'}{2} = 4° 32' 47''.$$
$$\therefore c' = 9° \ 5' 34''.$$

Ans. $B = 64° 26' \ 4''$, $C = 77° 21' 36''$, $c = 53° 35' 20''$;
$B' = 115° 35' 56''$, $C' = 11° 11' 40''$, $c' = 9° \ 5' 34''$.

Otherwise thus: Let fall the perpendicular CD; denote AD by m, the angle ACD by ϕ, and CD by p. Then we have



$$\cos A = \tan m \cot b;$$
$$\therefore \tan m = \cos A \tan \mathrm{b} \qquad (1)$$
$$\cos b = \cot A \cot \phi. \quad \therefore \cot \phi = \cos b \tan A \quad . \quad (2)$$

Again,

$$\cos a = \cos (c-m) \cos p; \ \cos b = \cos m \cos p.$$
$$\therefore \cos (c-m) = \cos a \cos m \div b \quad . \ . \ . \quad (3)$$

Also, $\cos (C-\phi) = \cot a \tan p$; $\cos \phi = \cot b \tan p$.

$$\therefore \cos (C-\phi) = \cot a \tan b \cos \phi \quad . \ . \ . \quad (4)$$

Lastly, $$\sin B = \frac{\sin b}{\sin a} \sin A \quad . \ . \ . \ . \ . \ . \quad (5)$$

The required parts are given by (1), (2), (3), (4), (5).

Ex. 2. Given $a = 73° 49' 38''$, $b = 120° 53' 35''$, $A = 88° 52' 42''$: find B, C, c.

Ans. $B = 116° 44' 48''$, $C = 116° 44' 48''$, $c = 120° 55' 35''$.

112. Case IV. — Given two angles, A, B, and the side a opposite one of them; to find b, c, C.

This case reduces, by aid of the polar triangle, to the preceding case, and gives rise to the same ambiguities. Hence the same remarks made in Art. 111 apply in this case also, and the direct solution may be obtained in the same way as in Case III. Thus,

The side b is found from the formula

$$\sin b = \frac{\sin \mathrm{B}}{\sin \mathrm{A}} \sin a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then c and C are found from Napier's Analogies.

$$\tan \frac{c}{2} = \frac{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})}{\cos \frac{1}{2}(\mathrm{A}-\mathrm{B})} \tan \tfrac{1}{2}(a+b) \quad . \quad . \quad . \quad (2)$$

$$\tan \frac{\mathrm{C}}{2} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) \quad . \quad . \quad . \quad (3)$$

Check, $\dfrac{\sin \mathrm{A}}{\sin a} = \dfrac{\sin \mathrm{B}}{\sin b} = \dfrac{\sin \mathrm{C}}{\sin c}$.

Ex. 1. Given $\mathrm{A} = 66° 7' 20''$, $\mathrm{B} = 52° 50' 20''$, $a = 59° 28' 27''$; find b, c, C.

Solution.

By (1) we find $b = 48° 39' 16''$, or $131° 20' 44''$.

$$\begin{aligned}
\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) &= 59° 28' 50''.\\
\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) &= 6° 38' 30''.\\
\log \cos \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) &= 9.70572\\
\operatorname{colog} \cos \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) &= 0.00292\\
\log \tan \tfrac{1}{2}(a+b) &= 10.13976\\
\log \tan \frac{c}{2} &= 9.84840\\
\therefore \frac{c}{2} &= 35° 11' 50''.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(a+b) &= 54° \ 3' 51\tfrac{1}{2}''.\\
\tfrac{1}{2}(a-b) &= 5° 24' 35\tfrac{1}{2}''.\\
\log \cos \tfrac{1}{2}(a-b) &= 9.99806\\
\operatorname{colog} \cos \tfrac{1}{2}(a+b) &= 0.23145\\
\log \cot \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) &= 9.77049\\
\log \tan \frac{\mathrm{C}}{2} &= 10.00000\\
\therefore \frac{\mathrm{C}}{2} &= 45°.
\end{aligned}$$

The second value of b is inadmissible (see Rule of Art. 111), and therefore there is only one solution.

Ans. $b = 48° 39' 16''$, $c = 70° 23' 40''$, $\mathrm{C} = 90°$.

Ex. 2. Given $A = 110° 10'$, $B = 133° 18'$, $a = 147° 5' 32''$; find b, c, C.

Ans. $b = 155° 5' 18''$, $c = 33° 1' 45''$, $C = 70° 20' 50''$.

113. Case V. — Given the three sides, *a*, *b*, *c*; to find the angles.

The angles may be computed by any of the formulæ of Art. 95; but since an angle near 90° cannot be accurately determined by its sine, nor one near 0° by its cosine, neither of the first six formulæ can be used with advantage in all cases. The formulæ for the tangents however are accurate in all parts of the quadrant, and are therefore to be preferred for the solution of a triangle in which all three sides or all three angles are given.

By (7) of Art. 95 we have

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$$

$$= \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

Since the part under the radical is a symmetric function of the sides, it is in the formulæ for determining all three angles A, B, C, and when once calculated, it may be utilized in the calculation of each angle. For convenience in computation, denote this term by $\tan r$. Then

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

and (7), (8), (9) of Art. 95 become

$$\tan\frac{A}{2} = \frac{\tan r}{\sin(s-a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tan\frac{B}{2} = \frac{\tan r}{\sin(s-b)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\tan\frac{C}{2} = \frac{\tan r}{\sin(s-c)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Check. $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$.

Ex. 1. Given $a = 46° 24'$, $b = 67° 14'$, $c = 81° 12'$; find A, B, C.

Solution.

$a = 46° 24'$	$\log \sin (s-a) = 9.89060$
$b = 67° 14'$	$\log \sin (s-b) = 9.70137$
$c = 81° 12'$	$\log \sin (s-c) = 9.44602$
$2s = 194° 50'$	$\text{colog} \sin s = 0.00365$
$s = 97° 25'$,	$\log \tan^2 r = 9.04164$
$s - a = 51° 1'$,	$\log \tan r = 9.52082$.
[illegible] $= 30° 11'$,	
[illegible] $= 16° 13'$.	

[illegible]		
[illegible]	$\tan r = 9.52082$	$\tan r = 9.52082$
[illegible]	$\sin (s-b) = 9.70137$	$\sin (s-c) = 9.44602$
[illegible]	$\tan \frac{B}{2} = 9.81945$	$\tan \frac{C}{2} = 10.07480$
[illegible]	$\therefore \frac{B}{2} = 33° 25' 10''$.	$\therefore \frac{C}{2} = 49° 54' 35''$.
[illegible]	$B = 66° 50' 20''$.	$C = 99° 49' 10''$.

[illegible] $= 100°$, $b = 37° 18'$, $c = 62° 46'$; find A, [illegible]

[illegible] $15' 46''$, $B = 2° 17' 55''$, $C = 3° 22' 25''$.

[illegible] **Given the three angles,** A, B, C; **to find** [illegible]

[illegible] formulæ for the tangents are to be [illegible]

$$[illegible] \sqrt{\frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)}},$$

[illegible] (9) of Art. 96,

[illegible] $\tan R \cos (S - A)$,

$$\tan\frac{b}{2} = \tan R \cos(S - B),$$

$$\tan\frac{c}{2} = \tan R \cos(S - C),$$

by which the three sides may be found.

Check, $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Ex. 1. Given $A = 68° 30'$, $B = 74° 20'$, $C = 83° 10'$; find a, b, c.

Solution.

$$\begin{array}{rl} A = & 68° \, 30' \\ B = & 74° \, 20' \\ C = & 83° \, 10' \\ \hline 2S = & 226° \;\; 0' \end{array} \qquad \begin{array}{rl} S = & 113° \;\; 0' \\ S - A = & 44° \, 30' \\ S - B = & 38° \, 40' \\ S - C = & 29° \, 50' \end{array}$$

$$\begin{array}{rl} \log(-\cos S) = & 9.59188 \\ \log\cos(S - A) = & 9.85324 \\ \log\cos(S - B) = & 9.89254 \\ \log\cos(S - C) = & \underline{9.93826} \\ \log\tan^2 R = & 9.90784 \\ \log\tan R = & 9.95392.\text{[*]} \end{array} \quad \Bigg| \quad \begin{array}{rl} \log\tan\frac{a}{2} = & 9.80716 \\ \log\tan\frac{b}{2} = & 9.84646 \\ \log\tan\frac{c}{2} = & 9.89218 \\ \hline a = & 65° \, 21' \, 22'' \\ b = & 70° \;\; 9' \, 10'', \\ c = & 75° \, 55' \;\; 9''. \end{array}$$

Check, $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Ex. 2. Given $A = 59° 55' 10''$, $B = 85° 36' 50°$, $C = 59° 55' 10''$; find a, b, c.

Ans. $a = 129° \, 11' \, 40''$, $b = 63° \, 15' \, 12''$, $c = 129° \, 11' \, 40''$.

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[*] The necessary additions may be conveniently performed by writing $\log\tan R$ on a slip of paper, and holding it successively over $\log\cos(S - A)$, $\log\cos(S - B)$, and $\log\cos(S - C)$.

EXAMPLES.

Solve the following right triangles:

1.	Given $c=84^\circ 20'$,	$A=35^\circ 25'$;	
	find $a=35^\circ 13' 4''$,	$b=83^\circ 3' 29''$,	$B=85^\circ 59' 1''$.
2.	Given $c=67^\circ 54'$,	$A=43^\circ 28'$;	
	find $a=39^\circ 35' 51''$,	$b=60^\circ 46' 25''$,	$B=70^\circ 22' 21''$.
3.	Given $c=22^\circ 18' 30''$,	$A=47^\circ 39' 36''$;	
	find $a=16^\circ 17' 41''$,	$b=15^\circ 26' 53''$,	$B=44^\circ 33' 53''$.
4.	Given $c=145^\circ$,	$A=23^\circ 28'$;	
	find $a=13^\circ 12' 12''$,	$b=147^\circ 17' 15''$,	$B=109^\circ 34' 33''$.
5.	Given $c=98^\circ 6' 43''$,	$A=138^\circ 27' 18''$;	
	find $a=137^\circ 6'$,	$b=77^\circ 51'$,	$B=80^\circ 55' 27''$.
6.	Given $c=46^\circ 40' 12''$,	$A=37^\circ 46' 9''$;	
	find $a=26^\circ 27' 23''$,	$b=39^\circ 57' 41''$,	$B=62^\circ 0' 4''$.
7.	Given $c=76^\circ 42'$,	$a=47^\circ 18'$;	
	find $b=70^\circ 10' 13''$,	$A=49^\circ 2' 24''$,	$B=75^\circ 9' 24''$.
8.	Given $c=91^\circ 18'$,	$a=72^\circ 27'$;	
	find $b=94^\circ 18' 53''$,	$A=72^\circ 29' 48''$,	$B=94^\circ 6' 53''$.
9.	Given $c=86^\circ 51'$,	$b=18^\circ 1' 50''$;	
	find $a=86^\circ 41' 14''$,	$A=88^\circ 58' 25''$,	$B=18^\circ 3' 32''$.
10.	Given $c=23^\circ 49' 51''$,	$a=14^\circ 16' 35''$;	
	find $b=19^\circ 17'$,	$A=37^\circ 36' 49''$,	$B=54^\circ 49' 23''$.
11.	Given $c=97^\circ 13' 4''$,	$a=132^\circ 14' 12''$;	
	find $b=79^\circ 13' 38''$,	$A=131^\circ 43' 50''$,	$B=81^\circ 58' 53''$.
12.	Given $c=37^\circ 40' 20''$,	$a=37^\circ 40' 12''$;	
	find $b=0^\circ 26' 37''$,	$A=89^\circ 25' 37''$,	$B=0^\circ 43' 33''$.

13.	Given $a = 82^\circ\ 6'$,	$B = 43^\circ 28'$;	
	find $A = 84^\circ 34' 28''$,	$b = 43^\circ 11' 38''$,	$c = 84^\circ 14' 57''$.
14.	Given $a = 42^\circ 30' 30''$,	$B = 53^\circ 10' 30''$;	
	find $A = 53^\circ 50' 12''$,	$b = 42^\circ\ 3' 47''$,	$c = 56^\circ 49'\ 8''$.
15.	Given $a = 20^\circ 20' 20''$,	$B = 38^\circ 10' 10''$;	
	find $A = 54^\circ 35' 16''$,	$b = 15^\circ 16' 50''$,	$c = 25^\circ 14' 38''$.
16.	Given $a = 92^\circ 47' 32''$;	$B = 50^\circ\ 2'\ 1''$;	
	find $A = 92^\circ\ 8' 23''$,	$b = 50^\circ$,	$c = 91^\circ 47' 40''$.
17.	Given $b = 54^\circ 30'$,	$A = 35^\circ 30'$;	
	find $B = 70^\circ 17' 35''$,	$a = 30^\circ\ 8' 39''$,	$c = 59^\circ 51' 20''$.
18.	Given $b = 155^\circ 46' 42''$,	$A = 80^\circ 10' 30''$;	
	find $B = 153^\circ 58' 24''$,	$a = 67^\circ\ 6' 52''$,	$c = 110^\circ 46' 20''$.
19.	Given $a = 35^\circ 44'$,	$A = 37^\circ 28'$;	
	find $b = 69^\circ 50' 24''$,	$c = 73^\circ 45' 15''$,	$B = 77^\circ 54'$,
	or $b' = 110^\circ\ 9' 36''$,	$c' = 106^\circ 14' 45''$,	$B' = 102^\circ\ 6'$.
20.	Given $a = 129^\circ 33'$,	$A = 104^\circ 59'$;	
	find $b = 18^\circ 54' 38''$,	$c = 127^\circ\ 2' 27''$,	$B = 23^\circ 57' 19''$,
	or $b' = 161^\circ\ 5' 22''$,	$c' = 52^\circ 57' 33''$,	$B' = 156^\circ\ 2' 41''$.
21.	Given $a = 21^\circ 39'$,	$A = 42^\circ 10' 10''$;	
	find $b = 25^\circ 59' 28''$,	$c = 33^\circ 20' 13''$,	$B = 52^\circ 23'\ 3''$,
	or $b' = 154^\circ\ 0' 32''$,	$c' = 146^\circ 39' 47''$,	$B' = 127^\circ 36' 57''$.
22.	Given $a = 42^\circ 18' 45''$,	$A = 46^\circ 15' 25''$;	
	find $b = 60^\circ 36' 10''$,	$c = 68^\circ 42' 59''$,	$B = 69^\circ 13' 47''$,
	or $b' = 119^\circ 23' 50''$,	$c' = 111^\circ 17'\ 1''$,	$B' = 110^\circ 46' 13''$.
23.	Given $b = 160^\circ$,	$B = 150^\circ$;	
	find $a = 39^\circ\ 4' 51''$,	$c = 136^\circ 50' 23''$,	$A = 67^\circ\ 9' 43''$,
	or $a' = 140^\circ 55'\ 9''$,	$c' = 43^\circ\ 9' 37''$,	$A' = 112^\circ 50' 17''$.

24.	Given $a = 25^\circ 18' 45''$,	$A = 15^\circ 58' 15''$.	
	***Ans.* Impossible; why?**		
25.	Given $a = 32^\circ 9' 17''$,	$b = 32^\circ 41'$;	
	find $A = 49^\circ 20' 17''$,	$B = 50^\circ 19' 16''$,	$c = 44^\circ 33' 17''$.
26.	Given $a = 55^\circ 18'$,	$b = 39^\circ 27'$;	
	find $A = 66^\circ 15' 6''$,	$B = 45^\circ 1' 31''$,	$c = 63^\circ 55' 21''$.
27.	Given $a = 56^\circ 20'$,	$b = 78^\circ 40'$;	
	find $A = 56^\circ 51' 7''$,	$B = 80^\circ 31' 48''$,	$c = 83^\circ 44' 44''$.
28.	Given $a = 86^\circ 40'$,	$b = 32^\circ 40'$;	
	find $A = 88^\circ 11' 57''$,	$B = 32^\circ 42' 37''$,	$c = 87^\circ 11' 39''$.
29.	Given $a = 37^\circ 48' 12''$,	$b = 59^\circ 44' 16''$;	
	find $A = 41^\circ 55' 45''$,	$B = 70^\circ 19' 15''$,	$c = 66^\circ 32' 6''$.
30.	Given $a = 116^\circ$,	$b = 16^\circ$;	
	find $A = 97^\circ 39' 24''$,	$B = 17^\circ 41' 39''$,	$c = 114^\circ 55' 20''$.
31.	Given $A = 52^\circ 26'$,	$B = 49^\circ 15'$;	
	find $a = 36^\circ 24' 34''$,	$b = 34^\circ 33' 40''$,	$c = 48^\circ 29' 20''$.
32.	Given $A = 64^\circ 15'$,	$B = 48^\circ 24'$;	
	find $a = 54^\circ 28' 53''$,	$b = 42^\circ 30' 47''$,	$c = 64^\circ 38' 38''$.
33.	Given $A = 54^\circ 1' 15''$,	$B = 137^\circ 24' 21''$;	
	find $a = 29^\circ 46' 8''$,	$b = 155^\circ 27' 55''$,	$c = 142^\circ 9' 12''$.
34.	Given $A = 46^\circ 59' 42''$,	$B = 57^\circ 59' 17''$;	
	find $a = 36^\circ 27'$,	$b = 43^\circ 32' 37''$,	$c = 54^\circ 20' 3''$.
35.	Given $A = 55^\circ 32' 45''$,	$B = 101^\circ 47' 56''$;	
	find $a = 54^\circ 41' 35''$,	$b = 104^\circ 21' 28''$,	$c = 98^\circ 14' 24''$.
36.	Given $A = 60^\circ 27' 24''$,	$B = 57^\circ 16' 20''$;	
	find $a = 54^\circ 32' 32''$,	$b = 51^\circ 43' 36''$,	$c = 68^\circ 56' 28''$.

Solve the following quadrantal triangles:

37.	Given $B= 74^\circ 45'$,	$a= 18^\circ 12'$,	$c= 90^\circ$;
	find $b= 85^\circ 17' 15''$,	$A= 17^\circ 34'\ 2''$,	$C=104^\circ 31' 13''$.
38.	Given $A=110^\circ 47' 50''$,	$B=135^\circ 35' 34''$,	$c= 90^\circ$;
	find $a=104^\circ 53'\ 0''$,	$b=133^\circ 39' 47''$,	$C=104^\circ 41' 37''$.

Solve the following oblique triangles:

39.	Given $a= 73^\circ 58'$,	$b= 38^\circ 45'$,	$C= 46^\circ 33' 39''$;
	find $A=116^\circ\ 8' 28''$,	$B= 35^\circ 46' 39''$,	$c= 51^\circ\ 1' 11''$.
40.	Given $a= 96^\circ 24' 30''$,	$b= 68^\circ 27' 26''$,	$C= 84^\circ 46' 40''$;
	find $A= 97^\circ 53' 0''$,	$B= 67^\circ 59' 39''$,	$c= 87^\circ 31' 37''$.
41.	Given $a= 76^\circ 24' 40''$,	$b= 58^\circ 18' 36''$,	$C=116^\circ 30' 28''$;
	find $A= 63^\circ 48' 35''$,	$B= 51^\circ 46' 12''$,	$c=104^\circ 13' 27''$.
42.	Given $a= 86^\circ 18' 40''$,	$b= 45^\circ 36' 20''$,	$C=120^\circ 46' 30''$;
	find $A= 64^\circ 48' 53''$,	$B= 40^\circ 23' 15''$,	$c=108^\circ 39' 11''$.
43.	Given $a= 88^\circ 24'$,	$b= 56^\circ 48'$,	$C=128^\circ 16'$;
	find $A= 65^\circ 13'\ 3''$,	$B= 49^\circ 27' 51''$,	$c=120^\circ 10' 52''$.
44.	Given $a= 68^\circ 20' 25''$,	$b= 52^\circ 13' 15''$,	$C=117^\circ 12' 20''$;
	find $A= 56^\circ 16' 15''$,	$B= 45^\circ\ 4' 41''$,	$c= 96^\circ 20' 44''$.
45.	Given $a= 88^\circ 12' 20''$,	$b=124^\circ\ 7' 17''$,	$C= 50^\circ\ 2'\ 1''$;
	find $A= 63^\circ 15' 12''$,	$B=132^\circ 17' 59''$,	$c= 59^\circ\ 4' 25''$.
46.	Given $A= 31^\circ 34' 26''$,	$B= 30^\circ 28' 12''$,	$c= 70^\circ\ 2'\ 3''$;
	find $a= 40^\circ\ 1'\ 5''$,	$b= 38^\circ 31'\ 3''$,	$C=130^\circ\ 3' 50''$.
47.	Given $A= 96^\circ 46' 30''$,	$B= 84^\circ 30' 20''$,	$c=126^\circ 46'$;
	find $a=102^\circ 21' 42''$,	$b= 78^\circ 17'\ 2''$,	$C=125^\circ 28' 13''$.
48.	Given $A=107^\circ 47'\ 7''$,	$B= 38^\circ 58' 27''$,	$c= 51^\circ 41' 14''$;
	find $a= 70^\circ 20' 50''$,	$b= 38^\circ 27' 59''$,	$C= 52^\circ 29' 45''$.

49.	Given $A=128^\circ 41' 49''$,	$B=107^\circ 33' 20''$,	$c=124^\circ 12' 31''$;
	find $a=125^\circ 44' 44''$,	$b=82^\circ 47' 35''$,	$C=127^\circ 22' 7''$.
50.	Given $A=129^\circ 58' 30''$,	$B=34^\circ 29' 30''$,	$c=50^\circ 6' 20''$;
	find $a=85^\circ 59'$,	$b=47^\circ 29' 20''$,	$C=36^\circ 6' 50''$.
51.	Given $A=95^\circ 38' 4''$,	$C=97^\circ 26' 29''$,	$b=64^\circ 23' 15''$;
	find $a=99^\circ 40' 48''$,	$c=100^\circ 49' 30''$,	$B=65^\circ 33' 10''$.
52.	Given $A=70^\circ$,	$B=131^\circ 18'$,	$c=116^\circ$;
	find $a=57^\circ 56' 53''$,	$b=137^\circ 20' 33''$,	$C=94^\circ 48' 12''$.
53.	Given $a=52^\circ 45' 20''$,	$b=71^\circ 12' 40''$,	$A=46^\circ 22' 10''$;
	find $B=59^\circ 24' 16''$,	$C=115^\circ 39' 55''$,	$c=97^\circ 33' 18''$,
	or $B'=120^\circ 35' 44''$,	$C'=26^\circ 59' 55''$,	$c'=29^\circ 57' 10''$.
54.	Given $a=48^\circ 45' 40''$,	$b=67^\circ 12' 20''$,	$A=42^\circ 20' 30''$;
	find $B=55^\circ 39' 57''$,	$C=116^\circ 34' 18''$,	$c=93^\circ 8' 9''$,
	or $B'=124^\circ 20' 3''$,	$C'=24^\circ 32' 15''$,	$c'=27^\circ 37' 20''$.
55.	Given $a=46^\circ 20' 45''$,	$b=65^\circ 18' 15''$,	$A=40^\circ 10' 30''$;
	find $B=54^\circ 6' 19''$,	$C=116^\circ 55' 26''$,	$c=90^\circ 31' 46''$,
	or $B'=125^\circ 53' 41''$,	$C'=24^\circ 12' 53''$,	$c'=27^\circ 23' 14''$.
56.	Given $a=150^\circ 57' 5''$,	$b=134^\circ 15' 54''$,	$A=144^\circ 22' 42''$;
	find $B=120^\circ 47' 44''$,	$C=97^\circ 42' 55''$,	$c=55^\circ 42' 8''$,
	or $B'=59^\circ 12' 16''$,	$C'=29^\circ 9' 9''$,	$c'=23^\circ 57' 29''$.
57.	Given $a=50^\circ 45' 20''$,	$b=69^\circ 12' 40''$,	$A=44^\circ 22' 10''$;
	find $B=57^\circ 34' 51''$,	$C=115^\circ 57' 50''$,	$c=95^\circ 18' 16''$,
	or $B'=122^\circ 25' 9''$,	$C'=25^\circ 44' 31''$,	$c'=28^\circ 45' 5''$.
58.	Given $a=99^\circ 40' 48''$,	$b=64^\circ 23' 15''$,	$A=95^\circ 38' 4''$;
	find $B=65^\circ 33' 10''$,	$C=97^\circ 26' 29''$,	$c=100^\circ 49' 35''$.
	(No ambiguity ; why?)		
59.	Given $A=79^\circ 30' 45''$,	$B=46^\circ 15' 15''$,	$a=53^\circ 18' 20''$;
	find $b=36^\circ 5' 34''$,	$c=50^\circ 24' 57''$,	$C=70^\circ 55' 35''$.
	(No ambiguity ; why?)		

60.	Given $A = 73^\circ 11' 18''$,	$B = 61^\circ 18' 12''$,	$a = 46^\circ 45' 30''$;
	find $b = 41^\circ 52' 34''$,	$c = 41^\circ 35' 4''$,	$C = 60^\circ 42' 46''$.
	(Only one solution; why?)		
61.	Given $A = 46^\circ 30' 40''$,	$B = 36^\circ 20' 20''$,	$a = 42^\circ 15' 20''$;
	find $b = 33^\circ 18' 47''$,	$c = 60^\circ 32' 6''$,	$C = 110^\circ 3' 14''$.
	(Only one solution; why?)		
62.	Given $A = 36^\circ 20' 20''$,	$B = 46^\circ 30' 40''$,	$a = 42^\circ 15' 20''$;
	find $b = 55^\circ 25' 2''$,	$c = 81^\circ 27' 26''$,	$C = 119^\circ 22' 27''$,
	or $b' = 124^\circ 34' 58''$,	$c' = 162^\circ 34' 27''$,	$C' = 164^\circ 41' 55''$.
63.	Given $A = 52^\circ 50' 20''$,	$B = 66^\circ 7' 20''$,	$a = 59^\circ 28' 27''$;
	find $b = 81^\circ 15' 15''$,	$c = 110^\circ 10' 50''$,	$C = 119^\circ 43' 48''$,
	or $b' = 98^\circ 44' 45''$,	$c' = 138^\circ 45' 26''$,	$C' = 142^\circ 24' 59''$.
64.	Given $A = 61^\circ 37' 52''$,	$B = 139^\circ 54' 34''$,	$b = 150^\circ 17' 26''$;
	find $a = 42^\circ 37' 17''$,	$c = 129^\circ 41' 4''$,	$C = 89^\circ 54' 19''$,
	or $a' = 137^\circ 22' 43''$,	$c' = 19^\circ 58' 35''$,	$C' = 26^\circ 21' 17''$.
65.	Given $A = 70^\circ$,	$B = 120^\circ$,	$b = 80^\circ$.
	Ans. Impossible; why?		
66.	Given $a = 108^\circ 14'$,	$b = 75^\circ 29'$,	$c = 56^\circ 37'$;
	find $A = 123^\circ 53' 47''$,	$B = 57^\circ 46' 56''$,	$C = 46^\circ 51' 51''$.
67.	Given $a = 57^\circ 17'$,	$b = 20^\circ 39'$,	$c = 76^\circ 22'$;
	find $A = 21^\circ 1' 2''$,	$B = 8^\circ 38' 46''$,	$C = 155^\circ 31' 36''$.
68.	Given $a = 68^\circ 45'$,	$b = 53^\circ 15'$,	$c = 46^\circ 30'$;
	find $A = 94^\circ 52' 40''$,	$B = 58^\circ 5' 10''$,	$C = 50^\circ 50' 52''$.
69.	Given $a = 63^\circ 54'$,	$b = 47^\circ 18'$,	$c = 53^\circ 26'$;
	find $A = 86^\circ 30' 40''$,	$B = 54^\circ 46' 14''$,	$C = 63^\circ 12' 55''$.
70.	Given $a = 70^\circ 14' 20''$,	$b = 49^\circ 24' 10''$,	$c = 38^\circ 46' 10''$;
	find $A = 110^\circ 51' 16''$,	$B = 48^\circ 56' 4''$,	$C = 38^\circ 26' 48''$.
71.	Given $a = 124^\circ 12' 31''$,	$b = 54^\circ 18' 16''$,	$c = 97^\circ 12' 25''$;
	find $A = 127^\circ 22' 7''$,	$B = 51^\circ 18' 11''$,	$C = 72^\circ 26' 40''$.

72.	Given $a = 50^\circ 12' 4''$,	$b = 116^\circ 44' 48''$,	$c = 129^\circ 11' 42''$;
	find $A = 59^\circ 4' 25''$,	$B = 94^\circ 23' 10''$,	$C = 120^\circ 4' 50''$.
73.	Given $A = 86^\circ 20'$,	$B = 76^\circ 30'$,	$C = 94^\circ 40'$;
	find $a = 87^\circ 20' 28''$,	$b = 76^\circ 44' 2''$,	$c = 93^\circ 55' 31''$.
74.	Given $A = 96^\circ 45'$,	$B = 108^\circ 30'$,	$C = 116^\circ 15'$;
	find $a = 88^\circ 27' 49''$,	$b = 107^\circ 19' 52''$,	$c = 115^\circ 28' 13''$.
75.	Given $A = 78^\circ 30'$,	$B = 118^\circ 40'$,	$C = 93^\circ 20'$;
	find $a = 74^\circ 57' 46''$,	$b = 120^\circ 8' 49''$,	$c = 100^\circ 18' 11''$.
76.	Given $A = 57^\circ 50'$,	$B = 98^\circ 20'$,	$C = 63^\circ 40'$;
	find $a = 58^\circ 8' 19''$,	$b = 83^\circ 5' 36''$,	$c = 64^\circ 3' 20''$.
77.	Given $A = 129^\circ 5' 28''$,	$B = 142^\circ 12' 42''$,	$C = 105^\circ 8' 10''$;
	find $a = 135^\circ 49' 20''$,	$b = 144^\circ 37' 15''$,	$c = 60^\circ 4' 54''$.
78.	Given $A = 102^\circ 14' 12''$,	$B = 54^\circ 32' 24''$,	$C = 89^\circ 5' 46''$;
	find $a = 104^\circ 25' 8''$,	$b = 53^\circ 49' 25''$,	$c = 97^\circ 44' 18''$.
79.	Given $A = 20^\circ 9' 56''$,	$B = 55^\circ 52' 32''$,	$C = 114^\circ 20' 14''$;
	find $a = 20^\circ 16' 38''$,	$b = 56^\circ 19' 41''$,	$c = 66^\circ 20' 43''$.

80. If a, b, c are each $< \frac{\pi}{2}$, show that the greater angle *may* exceed $\frac{\pi}{2}$.

81. If a alone $> \frac{1}{2}\pi$, show that A *must* exceed $\frac{\pi}{2}$.

82. If a and b are each $> \frac{1}{2}\pi$, and $c < \frac{1}{2}\pi$, prove that:

(1) The greatest angle A *must* be $> \frac{1}{2}\pi$;
(2) B *may* be $> \frac{1}{2}\pi$;
(3) C may or may not be $< \frac{1}{2}\pi$.

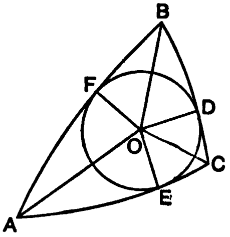
83. If $\cos a$, $\cos b$, $\cos c$ are all negative, prove that $\cos A$, $\cos B$, $\cos C$ are all necessarily negative.

CHAPTER VIII.

THE IN-CIRCLES AND CIRCUMCIRCLES.—AREAS.

115. The In-Circle (Inscribed Circle).—*To find the angular radius of the in-circle of a triangle.*

Let ABC be the triangle; bisect the angles A and B by the arcs AO, BO; from O draw OD, OE, OF perpendicular to the sides. Then it may be shown that O is the in-centre, and that the perpendiculars OD, OE, OF are each equal to the required angular radius.



Let $2s =$ the sum of the sides of the triangle ABC. The right triangles OAE, OAF are equal.

$$\therefore \text{AF} = \text{AE}.$$

Similarly, $\text{BD} = \text{BF}$, and $\text{CD} = \text{CE}$.

$$\therefore \text{BC} + \text{AF} = \text{AC} + \text{BF} = s.$$

$$\therefore \text{AF} = s - \text{BC} = s - a.$$

Now $\tan \text{OF} = \tan \text{OAF} \sin \text{AF}$. (Art. 86)

or, denoting the radius OF by r, we have

$$\tan r = \tan \frac{A}{2} \sin (s - a) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

or $$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}$$

$$= \frac{n}{\sin s} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(Art. 95)} \ (2)$$

Also, $\sin(s-a)$

$$= \sin\left[\tfrac{1}{2}(b+c) - \frac{a}{2}\right]$$

$$= \sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}a - \cos\tfrac{1}{2}(b+c)\sin\tfrac{1}{2}a$$

$$= \frac{\sin\frac{1}{2}a\cos\frac{1}{2}a}{\sin\frac{A}{2}}\left[\cos\tfrac{1}{2}(B-C) - \cos\tfrac{1}{2}(B+C)\right]. \quad \text{(Art. 98)}$$

$$= \frac{\sin a \sin\frac{1}{2}B \sin\frac{1}{2}C}{\sin\frac{1}{2}A},$$

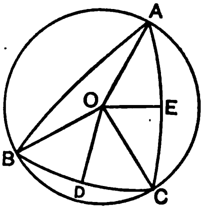
which in (1) gives

$$\tan r = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{1}{2}A}\sin a \quad \ldots\ldots\ldots \quad (3)$$

$$= \frac{N}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C} \quad \ldots \quad \text{(Art. 96)} \quad (4)$$

116. The Circumcircle. — *To find the angular radius of the circumcircle of a triangle.*

The small circle passing through the vertices of a spherical triangle is called the *circumscribing circle*, or *circumcircle*, of the triangle.



Let ABC be the triangle; bisect the sides CB, CA at D, E, and let O be the intersection of perpendiculars to CB, CA, at D, E; then O is the circumcentre.

For, join OA, OB, OC; then (Art. 86)

$$\cos OB = \cos BD \cos OD,$$

$$\cos OC = \cos DC \cos OD.$$

$$\therefore\ OB = OC. \quad \text{Similarly, } OC = OA.$$

Now the angle

$$OAB = OBA,\ OBC = OCB,\ OCA = OAC.$$

$$\therefore\ OCB + A = \tfrac{1}{2}(A+B+C) = S.$$

$$\therefore\ OCB = S - A.$$

Let OC = R; then, in the triangle ODC, we have

$$\cos \mathrm{OCD} = \tan \mathrm{CD} \cot \mathrm{CO} = \tan \tfrac{1}{2} a \cot \mathrm{R} \quad . \quad \text{(Art. 86)}$$

$$\therefore \tan \mathrm{R} = \frac{\tan \frac{1}{2} a}{\cos(\mathrm{S} - \mathrm{A})} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

or

$$\tan \mathrm{R} = -\frac{\cos \mathrm{S}}{\mathrm{N}} \quad . \; . \; . \; . \; . \; . \; . \; \text{(Art. 96)} \quad (2)$$

Also

$$\cos(\mathrm{S} - \mathrm{A}) = \cos \tfrac{1}{2}[(\mathrm{B} + \mathrm{C}) - \mathrm{A}]$$

$$= \cos \tfrac{1}{2}(\mathrm{B} + \mathrm{C}) \cos \tfrac{1}{2}\mathrm{A} + \sin \tfrac{1}{2}(\mathrm{B} + \mathrm{C}) \sin \tfrac{1}{2}\mathrm{A}$$

$$= \frac{\sin \frac{1}{2}\mathrm{A} \cos \frac{1}{2}\mathrm{A}}{\cos \frac{1}{2} a}[\cos \tfrac{1}{2}(b + c) + \cos \tfrac{1}{2}(b - c)] \quad \text{(Art. 98)}$$

$$= \frac{\sin \mathrm{A}}{\cos \frac{1}{2} a} \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c,$$

which in (1) gives

$$\tan \mathrm{R} = \frac{\sin \frac{1}{2} a}{\sin \mathrm{A} \cos \frac{1}{2} b \cos \frac{1}{2} c} \quad . \; . \; . \; . \; . \; . \quad (3)$$

$$= \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{n} \quad . \quad \text{(Art. 95)} \quad (4)$$

EXAMPLES.

1. Prove $\cos s + \cos(s - a) + \cos(s - b) + \cos(s - c)$
$= 4 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c.$

2. Prove $\cos(s - b) + \cos(s - c) - \cos(s - a) - \cos s$
$= 4 \cos \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c.$

AREAS OF TRIANGLES.

117. Problem. — *To find the area of a spherical triangle having given the three angles.*

Let r = the radius of the sphere.

E = the spherical excess = A + B + C − 180°.

K = area of triangle ABC.

It is shown in Geometry (Art. 738) that *the absolute area of a spherical triangle is to that of the surface of the sphere as its spherical excess, in degrees, is to 720°.*

$$\therefore\ K : 4\pi r^2 = E : 720°. \qquad \therefore\ K = \frac{E}{180°}\pi r^2 \quad . \ . \ . \quad (1)$$

118. Problem. — *To find the area of a triangle, having given the three sides.*

Here the object is to express E in terms of the sides.

I. *Cagnoli's Theorem.*

$$\sin\tfrac{1}{2}E = \sin\tfrac{1}{2}(A + B + C - \pi)$$

$$= \sin\tfrac{1}{2}(A + B)\sin\tfrac{1}{2}C - \cos\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}C$$

$$= \frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{\cos\frac{1}{2}c}\left[\cos\tfrac{1}{2}(a - b) - \cos\tfrac{1}{2}(a + b)\right] \quad . \ (\text{Art. } 98)$$

$$= \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin C}{\cos\frac{1}{2}c} = \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\cdot\frac{2n}{\sin a\sin b} \quad (\text{Art. } 95)\ (1)$$

$$\therefore\ \sin\tfrac{1}{2}E = \frac{n}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

II. *Lhuilier's Theorem.*

$$\tan\tfrac{1}{4}E = \frac{\sin\frac{1}{4}(A + B + C - \pi)}{\cos\frac{1}{4}(A + B + C - \pi)}$$

$$= \frac{\sin\frac{1}{2}(A + B) - \sin\frac{1}{2}(\pi - C)}{\cos\frac{1}{2}(A + B) + \cos\frac{1}{2}(\pi - C)} \quad . \ . \ . \ . \ . \ . \quad (\text{Art. } 41)$$

$$= \frac{\sin\frac{1}{2}(A + B) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(A + B) + \sin\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}(a - b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a + b) + \cos\frac{1}{2}c}\cdot\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C} \quad . \ . \ . \ . \ . \ . \quad (\text{Art. } 98)$$

$$= \frac{\sin\frac{1}{2}(s - b)\sin\frac{1}{2}(s - a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s - c)}\cot\tfrac{1}{2}C \quad . \ . \ . \ . \ . \quad (\text{Art. } 41)$$

$$= \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)} \quad (\text{Art. } 95)\ (3)$$

119. Problem. — *To find the area of a triangle, having given two sides and the included angle.*

$$\begin{aligned}\cos\tfrac{1}{2}\mathrm{E} &= \cos\left[\tfrac{1}{2}(\mathrm{A}+\mathrm{B})-(\tfrac{1}{2}\pi-\tfrac{1}{2}\mathrm{C})\right]\\ &= \cos\tfrac{1}{2}(\mathrm{A}+\mathrm{B})\sin\tfrac{1}{2}\mathrm{C}+\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B})\cos\tfrac{1}{2}\mathrm{C}\\ &= \cos\tfrac{1}{2}(a+b)\sin^2\tfrac{1}{2}\mathrm{C}+\cos\tfrac{1}{2}(a-b)\cos^2\tfrac{1}{2}\mathrm{C} \qquad \text{(Art. 98)}\\ &= [\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b+\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos\mathrm{C}]\sec\tfrac{1}{2}c \quad .\ . \quad (1)\end{aligned}$$

Dividing (1) of Art. 118 by this equation, and reducing, we have

$$\tan\tfrac{1}{2}\mathrm{E}=\frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin\mathrm{C}}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b\cos\mathrm{C}} \quad .\ .\ .\ .\ . \quad (2)$$

EXAMPLES.

1. Given $a=113°\,2'\,56''.64$, $b=82°\,39'\,28''.4$, $c=74°\,54'\,31''.06$; find the area of the triangle, the radius of the sphere being r.

By formula (3) of Art. 118,

$$\begin{aligned}a &= 113°\ \ 2'57''\\ b &= \ \ 82°39'28''\\ c &= \ \ 74°54'31''\\ \hline 2s &= 270°36'56''\\ \hline s &= 135°18'28''\\ s-a &= \ \ 22°15'31'',\\ s-b &= \ \ 52°39'\ \ 0'',\\ s-c &= \ \ 60°23'57''.\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}s &= 67°39'14''\\ \tfrac{1}{2}(s-a) &= 11°\ \ 7'46''\\ \tfrac{1}{2}(s-b) &= 26°19'30''\\ \tfrac{1}{2}(s-c) &= 30°11'58''\\ \hline \log\tan\tfrac{1}{2}s &= 0.38608\\ \log\tan\tfrac{1}{2}(s-a) &= 9.29386\\ \log\tan\tfrac{1}{2}(s-b) &= 9.69441\\ \log\tan\tfrac{1}{2}(s-c) &= 9.76492\\ \hline \log\tan^2\tfrac{1}{4}\mathrm{E} &= 9.13927\\ \log\tan\ \tfrac{1}{4}\mathrm{E} &= 9.56964.\\ \tfrac{1}{4}\mathrm{E} &= 20°\,21'\,58''.5.\\ \mathrm{E} &= 81°27'54''\\ &= 293274''.\end{aligned}$$

$$\therefore\ \mathrm{K}=\tfrac{293274}{648000}\times\pi r^2 \quad .\ \ .\ \ .\ \ . \quad [(1)\text{ of Art. }117]$$

$$=\tfrac{293274}{206265}r^2=\text{area.}$$

2. Given $A = 84°\,20'\,19''$, $B = 27°\,22'\,40''$, $C = 75°\,33'$;
find $E = 7°\,15'\,59''$.

3. Given $a = 46°\,24'$, $b = 67°\,14'$, $c = 81°\,12'$;
find $K = \frac{118380}{206265} r^2$.

4. Given $a = 108°\,14'$, $b = 75°\,29'$, $c = 56°\,37'$;
find $E = 48°\,32'\,34''$.

5. Prove

$$\cos\tfrac{1}{2}E = \frac{1 + \cos a + \cos b + \cos c}{4\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c}$$

$$= \frac{\cos^2\tfrac{1}{2}a + \cos^2\tfrac{1}{2}b + \cos^2\tfrac{1}{2}c - 1}{2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c}.$$

6. "

$$\sin\tfrac{1}{4}E = \sqrt{\frac{\sin\tfrac{1}{2}s\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)}{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c.}}.$$

7. "

$$\cos\tfrac{1}{4}E = \sqrt{\frac{\cos\tfrac{1}{2}s\cos\tfrac{1}{2}(s-a)\cos\tfrac{1}{2}(s-b)\cos\tfrac{1}{2}(s-c)}{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c}}.$$

EXAMPLES.

Prove the following:

1. $$\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s = 4\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c.$$

2. $$\sin s + \sin(s-b) + \sin(s-c) - \sin(s-a) = 4\sin\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c.$$

3. $$\sin^2 s + \sin^2(s-a) + \sin^2(s-b) + \sin^2(s-c) = 2(1 - \cos a\cos b\cos c).$$

4. $$\sin^2 s + \sin^2(s-a) - \sin^2(s-b) - \sin^2(s-c) = 2\cos a\sin b\sin c.$$

5. $$\cos^2 s + \cos^2(s-a) + \cos^2(s-b) + \cos^2(s-c) = 2(1 + \cos a\cos b\cos c).$$

6. $\cos^2 s + \cos^2(s-a) - \cos^2(s-b) - \cos^2(s-c)$
$= -2\cos a \sin b \sin c.$

7. In an equilateral triangle, $\tan R = 2\tan r$.

8. Given $a = 108° 14'$, $b = 75° 29'$, $c = 56° 37'$;
find $E = 48° 32' 34''.5$.

9. Given $a = 63° 54'$, $b = 47° 18'$, $c = 53° 26'$;
find $E = 24° 29' 49\frac{1}{2}''$.

10. Given $a = 69° 15' 6''$, $b = 120° 42' 47''$, $c = 159° 18' 33''$;
find $E = 216° 40' 23''$.

11. Given $a = 33° 1' 45''$, $b = 155° 5' 18''$, $C = 110° 10'$;
find $E = 133° 48' 55''$.

12. Given $a = b = c = 1°$, on the earth's surface;
find $E = 27''.21$.

13. Given $a = b = c = 60°$, on a sphere of 6 inches radius; find the area of the triangle. *Ans.* 19.845 square inches.

14. If $a = b = \frac{\pi}{3}$, and $c = \frac{\pi}{2}$, prove $\sin\frac{1}{2}E = \frac{1}{3}$, and $\cos E = \frac{7}{9}$.

15. If $C = \frac{\pi}{2}$, prove $\sin\frac{1}{2}E = \sin\frac{1}{2}a \sin\frac{1}{2}b \sec\frac{1}{2}c$, and $\cos\frac{1}{2}E = \cos\frac{1}{2}a \cos\frac{1}{2}b \sec\frac{1}{2}c$.

16. If $a = b$, and $C = \frac{\pi}{2}$, prove $\tan E = \frac{1}{2}\tan a \sec a$.

17. If $A + B + C = 2\pi$, prove $\cos^2\frac{1}{2}a + \cos^2\frac{1}{2}b + \cos^2\frac{1}{2}c = 1$.

18. If $a + b = \pi$, prove that $E = C$; and if E' denote the spherical excess of the polar triangle, prove that

$$\sin\tfrac{1}{2}E' = \sin a \cos\tfrac{1}{2}C.$$

120. Table of Formulæ.

PLANE TRIGONOMETRY.

1. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ . . . . (Art. 55)

2. $a^2 = b^2 + c^2 - 2bc \cos A$ . . . . (Art. 56)

3. $\dfrac{a+b}{a-b} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$ . . . . (Art. 57)

$$\left.\begin{array}{ll} 4. & \sin \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}} \\[2ex] 5. & \cos \frac{1}{2} A = \sqrt{\dfrac{s(s-a)}{bc}} \\[2ex] 6. & \tan \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \end{array}\right\} \text{. . . . . . . (Art. 59)}$$

7. $\sin A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$ . . . (Art. 60)

$= \dfrac{1}{2bc}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}.$

8. Area of $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ . . (Art. 61)

$$\left.\begin{array}{ll} 9. & \text{Area of } \triangle = \dfrac{r}{2}(a+b+c) = rs \\[2ex] 10. & r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}} \end{array}\right\} \text{. . (Art. 62)}$$

11. $R = \dfrac{abc}{4S}$ . . . . (Art. 63)

For other formulæ of Plane Trigonometry, see p. 63.

SPHERICAL TRIGONOMETRY.

$$\left.\begin{array}{ll} 12. & \cos c = \cos a \cos b \\ 13. & \sin b = \sin B \sin c \\ 14. & \sin a = \sin A \sin c \end{array}\right\} \text{. . . . . (Art. 85)}$$

$$\left.\begin{aligned} &15.\quad \cos C = -\cos A \cos B\\ &16.\quad \sin B = \sin b \sin C\\ &17.\quad \sin A = \sin a \sin C \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (\text{Art. } 89)$$

$$18.\quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (\text{Art. } 90)$$

$$\left.\begin{aligned} &19.\quad \cos a = \cos b \cos c + \sin b \sin c \cos A\\ &20.\quad \cos b = \cos c \cos a + \sin c \sin a \cos B\\ &21.\quad \cos c = \cos a \cos b + \sin a \sin b \cos C \end{aligned}\right\} \quad . \ . \quad (\text{Art. } 91)$$

$$\left.\begin{aligned} &22.\quad \cos A = -\cos B \cos C + \sin B \sin C \cos a\\ &23.\quad \cos B = -\cos C \cos A + \sin C \sin A \cos b\\ &24.\quad \cos C = -\cos A \cos B + \sin A \sin B \cos c \end{aligned}\right\} \quad (\text{Art. } 192)$$

$$\left.\begin{aligned} &25.\quad \sin \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \quad . \ . \ . \quad (\text{Art. } 95)\\ &26.\quad \cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.\\ &27.\quad \tan \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.\\ &28.\quad \sin A = \frac{2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}{\sin b \sin c} \end{aligned}\right\}$$

$$\left.\begin{aligned} &29.\quad \sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}} \quad . \ . \ . \ . \quad (\text{Art. } 96)\\ &30.\quad \cos \tfrac{1}{2} a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}.\\ &31.\quad \tan \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}.\\ &32.\quad \sin a = \frac{2\sqrt{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)}}{\sin B \sin C}. \end{aligned}\right\}$$

$$\left.\begin{aligned}
&\text{33.}\quad \tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C\\
&\text{34.}\quad \tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C\\
&\text{35.}\quad \tan\tfrac{1}{2}(a+b)\ =\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c\\
&\text{36.}\quad \tan\tfrac{1}{2}(a-b)\ =\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c
\end{aligned}\right\}\ .\ .\ \text{(Art. 97)}$$

$$\left.\begin{aligned}
&\text{37.}\quad \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C\\
&\text{38.}\quad \sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C\\
&\text{39.}\quad \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C\\
&\text{40.}\quad \cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C
\end{aligned}\right\}\ \text{(Art. 98)}$$

$$\left.\begin{aligned}
&\text{41.}\quad \tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}\\
&\qquad\qquad\ =\frac{n}{\sin s}
\end{aligned}\right\}\ \text{(Art. 115)}$$

$$\text{42.}\quad \tan R=\frac{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}{n}\ .\ .\ .\ .\ .\ .\ \text{(Art. 116)}$$

$$\text{43.}\quad K=\text{area of }\triangle=\frac{E}{180}\pi r^2\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \text{(Art. 117)}$$

$$\left.\begin{aligned}
&\text{44.}\quad \sin\tfrac{1}{2}E=\frac{n}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}\quad .\ .\ .\ .\ \text{(Art. 118)}\\
&\text{45.}\quad \tan\tfrac{1}{4}E=\sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}
\end{aligned}\right\}$$

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